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ENHANCED STRENGTH MODELS FOR NOTCHED LAMINATES WITH FINITE OUTER BOUNDARIES

For the practical design of notched fibre reinforced components, methods that consider finite anisotropic plates with a cutout are of special interest. The notch-induced stress concentrations lead to critical strains and frequently initiate catastrophic failure of the component. By selective fibre reinforcement of the matrix, a redistribution of the stress peaks relevant to failure can be achieved. In the course of this, the fibre orientation, besides the notch geometry, plays a decisive role. An anisotropic plate with finite dimensions and a hole in its center will be used here to analytically model stress concentrations. Unlike the infinite plate, this problem comprises a doubly connected outer area. A solution method has been developed for stress concentration problems of fibre-reinforced compounds based on the method of complex-valued stress functions combined with conformal mappings. Using the solution methods developed here, the whole calculation procedure was modified and extended in such a way, that even the influence of a finite outer boundary of the plate can be described. The consideration of the outer boundary is of high importance for practical problems, because then the essential influence of the notch size in dependence of the material, geometry and loading parameters can be also determined.

Key words: **anisotropy, plates, notch, stress concentration, finite outer boundary, conformal mapping**

UDOSKONALONE MODELE WYTRZYMAŁOŚCIOWE PŁYT LAMINATOWYCH O SKOŃCZONYCH ROZMIARACH Z KARBAMI

Metody obliczeniowe, uwzględniające skończone wymiary płyt i obecność karbów mają szczególne znaczenie dla praktycznych zagadnień dotyczących konstruowania elementów z kompozytów włóknistych. Spiętrzenia naprężeń wywołane działaniem karbów prowadzą często do pojawienia się krytycznych odkształceń, inicjujących zniszczenie struktury nośnej. Poprzez celowe umocnienie osnowy włóknami możliwe jest osiągnięcie rozdziału lokalnie działających naprężeń i tym samym uniknięcie uszkodzenia. Fakt ten świadczy o tym, że oprócz cech geometrycznych karbu również kierunek ułożenia włókien odgrywa istotną rolę. Przedstawiono analityczny model powstawania spiętrzenia naprężeń w anizotropowej płycie o skończonych wymiarach z centralnie usytuowanym otworem. W celu uwzględnienia skończonych wymiarów płyty obszar zewnętrzny traktuje się jako podwójnie połączony. Opisana metoda obliczania spiętrzenia naprężeń na skutek działania karbu łączy zastosowanie zespolonych funkcji opisujących naprężenia z odwzorowaniem konforemnym. Metoda ta umożliwia modyfikację i rozszerzenie procedury obliczeniowej w celu uwzględnienia wpływu skończonych wymiarów płyty. Możliwość uwzględnienia brzegu jest szczególnie ważna w zagadnieniach praktycznych, gdyż pozwala ona na ocenę działania krabu w zależności od materiału, cech geometrycznych oraz parametrów obciążenia.

Słowa kluczowe: **anizotropia, płyty, karby, spiętrzenie naprężeń, skończony brzeg, odwzorowanie konforemne**

INTRODUCTION

As notches represent the most relevant points of failure in a construction, a calculation of the stress distribution around holes is essential for the design of fibre-reinforced materials. Especially in the case of anisotropic materials, the maximum stress concentration at the cutouts is considerably higher than in conventional isotropic materials. In fibre-reinforced materials, the stress distribution around holes is strongly dependent on the degree of anisotropy $E_{\parallel}/E_{\perp}$ as well as on the notch geometry and load parameters. The plane stress field around a notch of known geometry will be calculated by means of the method of conformal mapping and complex stress functions, based on the mathematical model of an infinite anisotropic plate with various shapes of the aperture. For some standard types of notches and load cases, the stress concentration factor as a function of various design parameters will be studied for fibre-reinforced materials used in lightweight design. Particularly, the following design parameters are considered: hole geometry, fibre/matrix combination, load cases and orientation of fibres.

Due to the fact that plane anisotropic structures with notches are a significant problem for technical applications, this topic has been investigated from different points of view by numerous scientists. A basic introduction to this issue is presented in [1]. Research works of Hufenbach et al. [2-5] deal with technically relevant fibre-reinforced structures, for which analytical as well

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as experimental studies have been carried out. Extended analytical solutions for aniso-tropic plates with notches which have contours other than an ellipse can be found at Hufenbach and Kroll [4, 6], who verified their results by numerical and experimental methods. Some general solutions for infinite plates are also given e.g. by Becker [7] for unsymmetrical laminates with an elliptical hole. Modified formulations for the treatment of notched areas under hygrothermal loads are developed by Kroll [8] and for unsymmetrical cases by Zhou [9]. A generalized plate equation for multi-layered composites is given by Lepper [10].

FUNDAMENTAL EQUATION

The material behaviour of anisotropic composite plates is described by an extended generalized Hooke's law in the form

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} + \begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_{xy} \end{pmatrix} T_\Delta \qquad (1)$$

Here ε_x, ε_y, γ_{xy} and σ_x, σ_y, τ_{xy} denote strains and stresses respectively, S_{ij} are extensional compliances and α_x, α_y, α_{xy} are thermal extension coefficients. This material law, the equilibrium conditions and one of the compatibility equations give the generalized equation of the stress problem for anisotropic materials

$$\begin{aligned} & S_{22}\frac{\partial^4 F}{\partial x^4} - 2S_{26}\frac{\partial^4 F}{\partial x^3 \partial y} + (2S_{12}+S_{66})\frac{\partial^4 F}{\partial x^2 \partial y^2} - 2S_{16}\frac{\partial^4 F}{\partial x\, \partial y^3} + \\ & + S_{11}\frac{\partial^4 F}{\partial y^4} = -\left[(S_{12}+S_{22})\frac{\partial^2 U}{\partial x^2} - (S_{16}+S_{26})\frac{\partial^2 U}{\partial x \partial y} + \right. \\ & \left. + (S_{12}+S_{12})\frac{\partial^2 U}{\partial y^2}\right] - \left(\alpha_y \frac{\partial^2 T}{\partial x^2} - \alpha_{xy}\frac{\partial^2 T}{\partial x\, \partial y}\alpha_x \frac{\partial^2 T}{\partial y^2}\right) \end{aligned} \qquad (2)$$

with $F(x,y)$ being Airy's stress function, which is introduced in the usual way as follows:

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} + U, \quad \sigma_y = \frac{\partial^2 F}{\partial x^2} + U, \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \qquad (3)$$

Assuming that no potential U of the internal forces exists, the particular solution of (2) requires only the distribution of the temperature to be known.

For the case of a stationary thermal conduction in the laminate plane (no heat flux perpendicular to the mid-plane, $\dot{q}_z = 0$), the temperature field can be derived from the generalized Fourier differential equation

$$\lambda_{11}\frac{\partial^2 T}{\partial x^2} + 2\lambda_{12}\frac{\partial^2 T}{\partial x \partial y} + \lambda_{22}\frac{\partial^2 T}{\partial y^2} = 0 \qquad (4)$$

with λ_{ij} as thermal conductivities.

The homogenous solution of the differential equation of the anisotropic plate, that follows for purely mechanical loading, can, therefore, be stated in the form

$$\begin{aligned} F &= 2\,\mathrm{Re}\left[F_1(z_1) + F_2(z_2)\right] = \\ &= 2\,\mathrm{Re}\left[F_1(x+\mu_1 y) + F_2(x+\mu_2 y)\right] \end{aligned} \qquad (5)$$

where $\mu_1 = \alpha_1 + i\beta_1$ and $\mu_2 = \alpha_2 + i\beta_2$ are two complex conjugated roots of the characteristic equation

$$S_{11}\mu^4 - 2S_{16}\mu^3 + (2S_{12}+S_{66})\mu^2 - 2S_{26}\mu + S_{22} = 0 \qquad (6)$$

with S_{ij} $(i, j = 1, 2, 6)$ as compliances.

The conformal mapping:

$$z_k = \omega_k(\zeta_k) = \frac{a - i\mu_k b}{2}\zeta_k + \frac{a + i\mu_k b}{2}\frac{1}{\zeta_k} \qquad (7)$$

$$(k = 1, 2)$$

- with a and b being the large or small semi-axis, respectively - maps the outside domain of the ellipse in the z_k plane to the outside domain of the unit circle in the ζ_k plane. To simplify the notation, the analytical functions $f_k(\zeta)_k$ are introduced as

$$f_k(\zeta_k) = F_k'(\omega(\zeta_k)) \quad (k = 1, 2) \qquad (8)$$

They are approximated by Laurent series cut off at a finite number n^* of series terms

$$f_k(\zeta_k) = A_{k0} + \sum_{m=1}^{n^*} A_{km}\zeta_k^m + \sum_{m=1}^{n^*} A_{km}^*\zeta_k^{-m} \qquad (9)$$

$$(k = 1, 2)$$

The stresses, displacements and boundary conditions are then expressed using the two analytical functions in the ζ_k planes:

$$\begin{aligned} \sigma_x &= 2\,\mathrm{Re}\left[\sum_{k=1}^{2}\mu_k^2 \frac{f_k'(\zeta_k)}{\omega_k(\zeta_k)}\right] \\ \sigma_y &= 2\,\mathrm{Re}\left[\sum_{k=1}^{2}\frac{f_k'(\zeta_k)}{\omega_k'(\zeta_k)}\right] \\ \tau_{xy} &= -2\,\mathrm{Re}\left[\sum_{k=1}^{2}\mu_k \frac{f_k'(\zeta_k)}{\omega_k'(\zeta_k)}\right] \end{aligned} \qquad (10)$$

$$u = 2\,\mathrm{Re}\left[\sum_{k=1}^{2} p_k f_k(\zeta_k)\right] + c_1 + c_3 y$$
$$v = 2\,\mathrm{Re}\left[\sum_{k=1}^{2} q_k f_k(\zeta_k)\right] + c_2 - c_3 x \qquad (11)$$

$$2\,\mathrm{Re}\left[\sum_{k=1}^{2} f_k(\zeta_k)\right] = -\int_0^S Y_n ds + c_1^*$$
$$2\,\mathrm{Re}\left[\sum_{k=1}^{2} \mu_k f_k(\zeta_k)\right] = \int_0^S X_n ds + c_2^* \qquad (12)$$

$$2\,\mathrm{Re}\left[\sum_{k=1}^{2} p_k f_k(\zeta_k)\right] + c_1 + c_3 y = u^*$$
$$2\,\mathrm{Re}\left[\sum_{k=1}^{2} q_k f_k(\zeta_k)\right] + c_2 - c_3 x = v^* \qquad (13)$$

In (11) and (13), the constants c_1, c_2 and c_3 describe the translational rigid body displacement as well as torsion with respect to an arbitrary fixed point. In (12) and (13), X_n and Y_n are the given edge stresses and u^* and v^* the edge displacements in the x and y directions, respectively (cf. [8-10]).

Beyond this, the particular functions $F_T(z_T)$, with $z_T = x + \mu_T y$ have to be considered also for the case of additional influence of heat. Here, the complex parameter μ_T is determined from the Fourier differential equation as a root of the characteristic equation

$$\lambda_{22}\mu_T^2 + 2\lambda_{12}\mu_T + \lambda_{11} = 0 \qquad (14)$$

The roots from (14) are complex conjugates, which can be seen from the transformation behavior of the (always positive) basic values of the thermal conductivities $\lambda_{ij} : \mu_{T1} = \mu_T,\ \mu_{T2} = \overline{\mu}_T$. As a general solution of the equation of plane stress for anisotropic materials under mechanical and thermal loading (here $U(x,y) = 0$), it follows for the real-valued Airy's stress function

$$F(x,y) = 2\,\mathrm{Re}\left[F_1(z_1) + F_2(z_2) + F_T(z_T)\right] \qquad (15)$$

If the anisotropic plate is additionally subjected to a centrifugal loading (e.g. for fibre reinforced disc rotors), the volume forces induced by a conservative force field can be determined from the potential $U = -\rho\omega^2\left(x^2 + y^2\right)/2$. In this case, another analytical function $F_U(z_U)$ has to be added to (15), see e.g. [11]. The stress can now be calculated from:

$$\sigma_x = 2\,\mathrm{Re}\left[\mu_1^2 F_1''(z_1) + \mu_2^2 F_2''(z_2) + \mu_T^2 F_T''(z_T)\right]$$
$$\sigma_y = 2\,\mathrm{Re}\left[F_1''(z_1) + F_2''(z_2) + F_T''(z_T)\right] \qquad (16)$$
$$\tau_{xy} = -2\,\mathrm{Re}\left[\mu_1 F_1''(z_1) + \mu_2 F_2''(z_2) + \mu_T F_T''(z_T)\right]$$

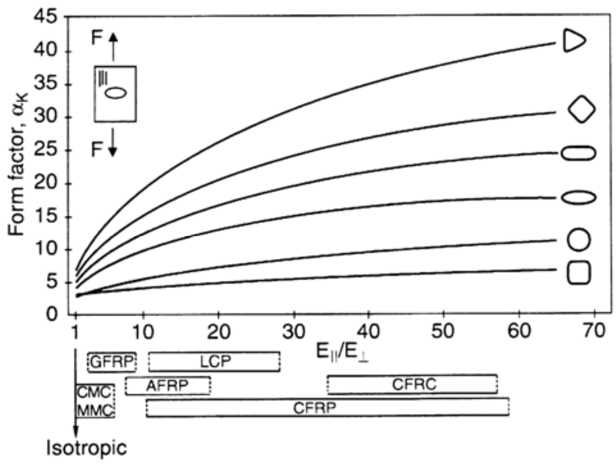
Therefore, the displacements u und v result from inserting the stress components in the material law (1) and subsequent integration:

$$u = 2\,\mathrm{Re}\left[p_1 F_1' + p_2 F_2' + p_T F_T'\right]$$
$$v = 2\,\mathrm{Re}\left[q_1 F_1' + q_2 F_2' + q_T F_T'\right] \qquad (17)$$

STRESS CONCENTRATION

The directional material data of anisotropic materials causes a direct influence of the elastic properties on the stress concentrations, whereas for isotropic materials stress concentrations are independent of the elastic properties. Theoretical and experimental investigations show that for a purely mechanical loading in fibre direction (on-axis loading) of technical composites the maximum stress concentrations strongly depend on the so-called degree of anisotropy $E_\parallel/E_\perp$. The indices $\parallel$ and $\perp$ denote the directions of the orthogonal material axes, where the index $\parallel$ indicates the direction of the higher modulus.

For plates under tensile loads in the direction of the highest Young's modulus, which have a relatively low degree of anisotropy, e.g. UD GFRP laminates in Figure 1, it has been found that even a small increase of the degree of anisotropy results in a very distinct gradient of stress concentration, whereas for „highly anisotropic" composites, e.g. UD CFRP laminates, the influence of the degree of anisotropy on the stress concentrations is a lot smaller (Fig. 1). A similar effect of the degree of anisotropy on the stress concentrations can be found for in-plane bending stresses.



CFRP Carbon-fibre reinforced polymer
AFRP Aramide-fibre reinforced polymer
GFRP Glass-fibre reinforced polymer
LCP Liquid crystal polymer
CFRC Carbon-fibre reinforced carbon
CMC Ceramic matrix composite
MMC Metal matrix composite

Fig. 1. Form factor dependent on the degree of anisotropy $E_{\parallel}/E_{\perp}$ for common unidirectional lightweight materials

The stress concentrations for a large number of presently used fibre-matrix composites have been analysed dependent on different notch shapes. The calculation results for composite plates with an infinite outer boundary under tensional loads are shown in Figure 1. These values have to be considered as the lower limit of the maximum stress concentrations and will rise as an effect of a finite outer boundary.

The influence of the geometry of the outer boundary and the fibre orientation on the stress concentrations is presented in Figure 2 for a CFRP-plate (CF-EP-T3.6 with E_1 = 138 GPa, E_2 = 8.5 GPa, ν_{12} = 0.29 and G_{12} = = 4.5 GPa) with an elliptical notch (a = 10 mm, b = 6 mm) loaded with an interior pressure of p = = 100 bar. The outer boundary is free. Within the presented theory, a plate with a notch can be regarded as a plate with a centric cutout. Figure 2 shows the ratio of the tangential stresses at the inner boundary of the finite and infinite plate σ_t and σ_{t0}, respectively, for $\theta = 0^o$ and $\theta = 90^o$ dependent on the ratio A/a (= B/b) of the semi-axes of the outer and inner boundary. Here, it can clearly be seen, that the influence of the boundary is significantly increasing for A/a < 10 dependent on the fibre orientation.

For the design of notched fibre-reinforced composites, the maximum stress concentration curves (Figs. 1-2) allow an easy and quick estimation of the main influencing factors dependent on the fibre-matrix combination.

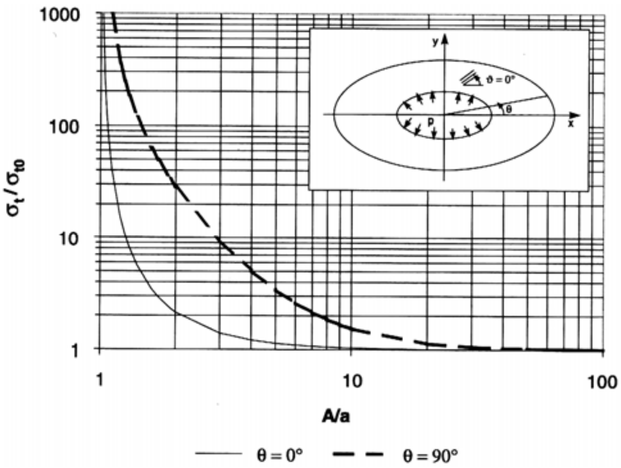
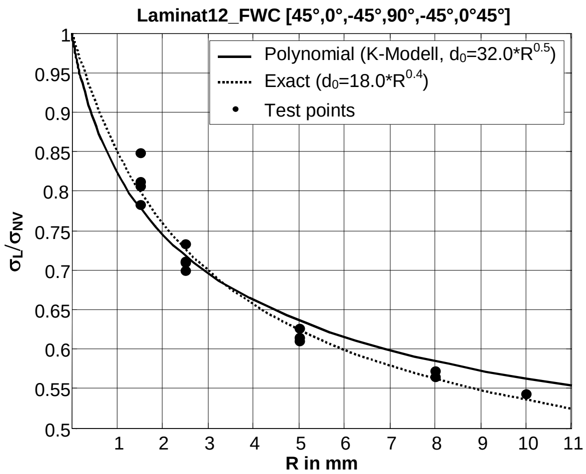


Fig. 2. Interaction of inner and outer boundary on stress concentration

STRENGTH ANALYSIS

Semi-empirical criteria developed in recent years for determining the strength of anisotropic plates with notches are based mainly on rough approximations of the stress concentration decay. Main assumptions are here e.g. polynomial shape of the decay function and transferability of the stress concentrations from an infinite to a finite plate [12-14]. Correction factors are usually derived from a stress analysis for isotropic materials. Semi-empirical failure models assume that a composite plate with a notched region fails exactly when the stress concentration at a characteristic distance from the edge of the notch reaches the strength of a notch-free composite. Very often, the experimentally measured characteristic distances are taken to be independent of laminate design, fibre orientation and dimensions of notches; they are then transferred to a great variety of laminates made of the same type of fibre. The disadvantage of such an analysis strategy is that physically inconsistent results are produced for some fibre orientations in highly anisotropic laminates. For this reason, such macroscopic failure models are applied for the strength analysis of quasi-isotropic laminates only. Another shortcoming of these failure criteria is that the exact place and mode of failure as distinctive failure characteristics are not taken into account.



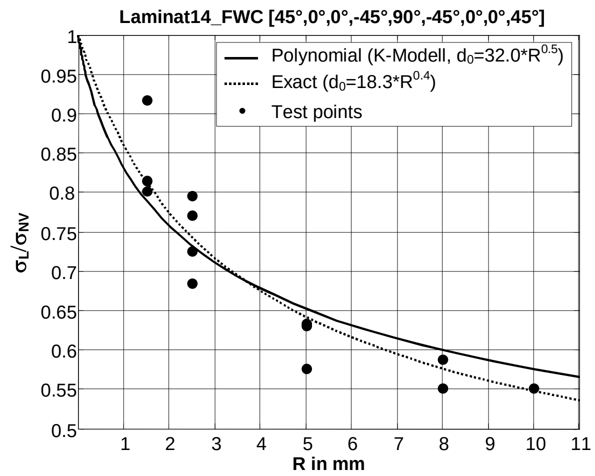


Fig. 3. Notched strength for typical aircraft CFRP laminates

The presented improved stress analysis of notched anisotropic plates combined with a physically based criterion gives the possibility of developing more accurate failure models, which allow a detailed and realistic prediction of the reliability on notched composite structures. Figure 3 exemplarily shows the theoretical notched strength versus the notch size for typical aircraft CFRP laminates (28.6, 57.1, 14.3) and (44.4, 44.4, 11.1) as well as the respective experimental results. In the graph, both the theoretical notched strength predicted by the conventional method (polynomial approximation of the stress concentration decay; classical Karlak approach for the notched strength [13, 14]) and the notched strength calculated with the presented improved method (exact stress concentration decay; modified failure hypothesis within the Karlak criterion) are shown. Thus, it can be seen that the improved model developed here describe the notched strength much more realistically than the conventional approach.

CONCLUSIONS

The improved stress analysis presented here is much more effective than purely numerical methods like a finite element analysis, because the new method makes it possible to quickly vary the different influencing variables. Especially for parameter studies concerning the influence of the outer and inner boundary contour on the stress concentrations, the lengthy and tedious generation of meshes is avoided. The presented calculation concept combined with a modified physically based criterion enables the engineer dealing with fibre composites to design critical areas already before the actual component dimensioning in such a way that the tailored load capacity of the structure fulfils all the objectives in a composite-suited manner.

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Recenzent
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