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VISCOELASTIC MODELLING OF REGULAR CROSS-PLY LAMINATES

The paper concerns regular cross-ply fibre-reinforced-plastic (CP xFRP) laminates, i.e., a stack of plies of $[\,/90]_{nS}$, $n \geq 4$ configuration. Each ply is a UD xFRP composite, i.e. an isotropic hardening plastic reinforced with long monotropic fibres packed unidirectionally in a hexagonal scheme. The plies are identical with respect to their thickness and microstructure. A polymer matrix of the laminate is an advanced viscoelastic isotropic material described by normal and fractional exponential functions, reflecting short-, moderate- and long-term viscoelastic processes. Fibres are made of a monotropic elastic material. The considerations are limited to stress levels protecting geometrically and physically linear viscoelastic behaviour of the material. The study presents a method for viscoelastic modelling of regular CP xFRP laminates, based on the exact stiffness theory of CP xFRP and on the elastic - viscoelastic analogy principle. Five independent elasticity compliances of a UD xFRP composite have been expressed in terms of the elasticity shear compliance of a viscoelastic isotropic polymer matrix. The elastic-viscoelastic analogy principle gives complex compliances of a UD xFRP composite dependent on complex shear compliance of the matrix. These compliances are used to determine complex compliances of a CP xFRP laminate. Standard constitutive equations of linear viscoelasticity of CP xFRP laminates are developed. The homogenized laminate is modelled as linearly viscoelastic orthotropic continuum described by 6 effective elasticity constants and 6 effective viscoelasticity coefficients. There are introduced the RLTC (relative long-term creep) coefficients dependent of micro- and meso-structure of the laminate as well as of the viscoelastic properties of the matrix, fully describing the standard viscoelasticity equations. A set of RLTC coefficients is calculated analytically. The method adopts the exact and approximate complex compliances of the laminate. A computer-aided algorithm presented in the study for calculation RLTC coefficients has been programmed in PASCAL. A regular CP CFRP laminate of $[\,/90]_{nS}$, $n \geq 4$ configuration, denoted with the symbol CP U/E53, has been examined as an example. Each ply is the UD U/E53 composite. The matrix (E53 hardening plastic) is made of Epidian 53 epoxide resin, reinforced with UTS 5631 carbon fibres produced by Tenax Fibers. Diagrams presenting the selected storage and loss compliances of the laminate are presented as well.

Keywords: regular cross-ply laminate, homogenization, constitutive equations of viscoelasticity, computer-aided algorithm

MODELOWANIE LEPKOSPRĘŻYSTE REGULARNYCH LAMINATÓW KRZYŻOWYCH

Praca dotyczy regularnych laminatów krzyżowych z matrycą duroplastyczną, tj. laminatów o konfiguracji warstw $[\,/90]_{nS}$, $n \geq 4$. Każda warstwa jest kompozytem wzmocnionym włóknem długim jednokierunkowo, równomiernie, w schemacie heksagonalnym. Warstwy są identyczne w zakresie grubości i mikrostruktury. Matryca jest zaawansowanym materiałem izotropowym lepkosprężystym, opisanym przez funkcje wykładnicze zwykłe i ułamkową, odwzorowujące procesy lepkosprężyste krótko-, średnio- i długotrwałe. Włókna są materiałem monotropowym sprężystym. Rozważania ograniczono do poziomów naprężeń gwarantujących geometrycznie i fizycznie liniowe zachowanie się materiału. Przedstawiono modelowanie lepkosprężyste rozpatrywanego typu laminatów, oparte na teorii sztywności regularnych laminatów krzyżowych oraz na analogii sprężystej - lepkosprężystej. Pięć niezależnych podatności sprężystych kompozytu wzmocnionego jednokierunkowo wyrażono przez podatność postaciową sprężystą izotropowej matrycy lepkosprężystej. Analogia sprężysta-lepkosprężysta pozwala na wyznaczenie podatności zespolonych tego kompozytu wyrażonych przez podatność zespoloną postaciową matrycy. Wymienione podatności zespolone wykorzystano do wyznaczenia podatności zespolonych regularnego laminatu krzyżowego. Sformułowano standardowe równania konstytutywne liniowej lepkosprężystości laminatów krzyżowych po homogenizacji. Laminat jest modelowany jako continuum ortotropowe lepkosprężyste opisane przez 6 efektywnych stałych sprężystości i 6 współczynników lepkosprężystości. Wprowadzono względne współczynniki pełzania długotrwałego, w pełni opisujące standardowe równania lepkosprężystości laminatu. Współczynniki te zależą od struktury mikro- i mezolaminatu oraz od właściwości lepkosprężystych matrycy. Wyznaczono je analitycznie, wykorzystując ścisłe i przybliżone podatności zespolone laminatu. Algorytm wyznaczania tych współczynników zaprogramowano w języku PASCAL. Jako przykład przedstawiono wyniki obliczeń w odniesieniu do laminatu CP U/E53 o konfiguracji $[\,/90]_{nS}$, $n \geq 4$. Matrycą jest duroplast E53 wytworzony z żywicy epoksydowej Epidian 53. Wzmocnienie każdej warstwy stanowią włókna UTS 5631 produkowane przez Tenax Fibers. Przedstawiono również wykresy części rzeczywistej i urojonej wybranej podatności zespolonej laminatu.

Słowa kluczowe: regularne laminaty krzyżowe, homogenizacja, równania konstytutywne lepkosprężystości, komputerowo wspomagany algorytm

INTRODUCTION

So far, a problem of viscoelastic modelling has been solved only for UD xFRP composites, i.e. hardening plastics reinforced with long fibres aligned unidirectionally [1, 2]. Viscoelastic modelling presented in Refs. [1, 2] employs the most advanced rheological model of chemically hardening plastics, capable of modelling short-, moderate- and long-term rheological processes.

On the micromechanics level, a UD xFRP composite is modelled as a linearly viscoelastic monotropic continuum with the monotropy axis coinciding the direction of fibres' alignment. The following assumptions are adopted:

– each ply is a two-phase material,
– both constituents, a matrix and a fibre, are homogeneous,
– stresses are restricted to the levels protecting linear behaviour of the constituents,
– there are considered quasi-static isothermal processes in the normal conditions, i.e. the processes belonging to the transition regime under the glass transition temperature,
– a matrix is a chemically hardening plastic made of a crosslinked polymer, modelled as a viscoelastic isotropic material, described by the rheological model presented in Ref. [3],
– a fibre is modelled as an elastic monotropic material (isotropic, in particular),
– fibres have identical solid circular cross-section; they are rectilinear and embedded uniformly in the matrix, in a hexagonal scheme,
– preparation of the fibres protects perfect bonding of the fibres to the matrix,
– residual stresses resulting from the manufacturing process are neglected,
– the Boltzmann superposition principle is obligatory.

Each ply (a UD xFRP composite) is described in the $x_1x_2x_3$ - Cartesian coordinate system with x_1 - a monotropy axis, and x_2x_3 - a transverse isotropy plane. The constituents are characterized by the following elasticity constants: E, ν (a Young's modulus, a Poisson's ratio of the matrix), $\overline{E}_1, \overline{E}_2, \overline{\nu}_{32}, \overline{\nu}_{21}, \overline{G}_{12}$ (longitudinal and transverse Young's moduli, Poisson's ratios in respective planes, a shear modulus in the monotropy plane of the fibre). The composite is also described by the fibre volume fraction *f*. Monotropic continuum modelling the homogenized ply is described by five independent effective elasticity constants (EECs), i.e., $E_1, E_2, \nu_{32}, \nu_{21}, G_{12}$ (effective longitudinal and transverse Young's moduli, effective Poisson's ratios in respective planes, an effective shear modulus in the monotropy plane). These constants are derived in terms of elasticity constants of the constituents and of the fibre volume fraction from the exact homogenization theory summarized in Ref. [4].

On the mesomechanics level, a regular CP xFRP laminate is modelled as a homogeneous orthotropic continuum, [5], described in the xyz - Cartesian coordinate system. Axes x, y coincide the lamination directions of respective groups of plies, whereas axis z is perpendicular to the xy midplane. For shortening, the symbols LEC and LVC are introduced, respectively denoting constitutive equations of linear elasticity and viscoelasticity.

Standard LEC equations of the homogenized regular CP xFRP laminate have the following well-known form [1-4]

$$\boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\sigma} \tag{1}$$

where

$$\begin{aligned} \boldsymbol{\sigma} &= \mathrm{col}\left(\sigma_x, \sigma_y, \sigma_z, \sigma_{yz}, \sigma_{xz}, \sigma_{xy}\right) \\ \boldsymbol{\varepsilon} &= \mathrm{col}\left(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}\right) \end{aligned} \tag{2}$$

are stress and strain vectors in the xyz - system, i.e.: $\sigma_x, \sigma_y, \sigma_z$ - normal stresses, $\sigma_{yz}, \sigma_{xz}, \sigma_{xy}$ - shear stresses, $\varepsilon_x, \varepsilon_y, \varepsilon_z$ - directional strains, $\gamma_{yz}, \gamma_{xz}, \gamma_{xy}$ - shear strains. The elasticity compliance matrix

$$\mathbf{S} = \begin{bmatrix} S_{11}^{\mathrm{cp}} & S_{12}^{\mathrm{cp}} & S_{13}^{\mathrm{cp}} & 0 & 0 & 0 \\ S_{12}^{\mathrm{cp}} & S_{22}^{\mathrm{cp}} & S_{23}^{\mathrm{cp}} & 0 & 0 & 0 \\ S_{13}^{\mathrm{cp}} & S_{23}^{\mathrm{cp}} & S_{33}^{\mathrm{cp}} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44}^{\mathrm{cp}} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55}^{\mathrm{cp}} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66}^{\mathrm{cp}} \end{bmatrix} \tag{3}$$

contains the following elasticity compliances

$$\begin{aligned} & S_{11}^{\mathrm{cp}} = \frac{1}{E_x}, \quad S_{22}^{\mathrm{cp}} = \frac{1}{E_y}, \quad S_{33}^{\mathrm{cp}} = \frac{1}{E_z} \\ & S_{12}^{\mathrm{cp}} = -\frac{\nu_{yx}}{E_x}, \quad S_{13}^{\mathrm{cp}} = -\frac{\nu_{zx}}{E_x}, \quad S_{23}^{\mathrm{cp}} = -\frac{\nu_{zy}}{E_y} \\ & S_{44}^{\mathrm{cp}} = \frac{1}{G_{yz}}, \quad S_{55}^{\mathrm{cp}} = \frac{1}{G_{xz}}, \quad S_{66}^{\mathrm{cp}} = \frac{1}{G_{xy}} \end{aligned} \tag{4}$$

expressed in terms of the EECs of the homogenized laminate, i.e. E_x, E_y, E_z - Young's moduli in the x, y, z directions, $\nu_{zy}, \nu_{zx}, \nu_{yx}$ - Poisson's ratios in respective planes, G_{yz}, G_{xz}, G_{xy} - shear moduli in respective planes. For a regular CP xFRP laminate only six EECs take different values, i.e. $E_x, E_z, \nu_{zx}, \nu_{yx}, G_{xz}, G_{xy}$.

PREDICTING EECs FOR A REGULAR CP xFRP LAMINATE

The exact homogenization theory of a regular CP xFRP laminate has been formulated in Ref. [5]. This theory gives the following final set of analytical formulae predicting EECs:

$$\begin{aligned} &E_x = E_y = \tfrac{1}{2} a\left(1-b^2\right), \quad E_z = \frac{a(1+b)}{a(1+b)d + c^2} \\ &\nu_{zx} = \nu_{zy} = \tfrac{1}{2} c(1-b), \quad \nu_{yx} = b \\ &G_{xz} = G_{yz} = \frac{2G_{12}G_{23}}{G_{12}+G_{23}}, \quad G_{xy} = G_{12} \end{aligned} \tag{5}$$

where

$$\begin{aligned} &a = \frac{E_1+E_2}{1-\nu_{21}\nu_{12}}, \quad b = \frac{E_1\nu_{12}+E_2\nu_{21}}{E_1+E_2} \\ &c = \frac{\nu_{21}(1+\nu_{12}+\nu_{32})+\nu_{32}}{1-\nu_{21}\nu_{12}} \\ &d = \frac{1-2\nu_{21}\nu_{12}(1+\nu_{32})-\nu_{32}^2}{E_2(1-\nu_{21}\nu_{12})} \end{aligned} \tag{6}$$

with

$$\nu_{12} = \frac{E_2}{E_1}\nu_{21}, \quad G_{23} = \frac{E_2}{2(1+\nu_{32})} \tag{7}$$

THE EXACT COMPLEX COMPLIANCES OF A CP xFRP LAMINATE

Complex compliances related to steady-state harmonic processes play a principal role in viscoelastic modelling of xFRP laminates [1, 2]. The exact complex compliances of the homogenized regular CP xFRP laminate can be calculated analytically via employing the elastic-viscoelastic analogy principle. It would be very advantageous to use the exact complex compliances of a UD xFRP composite obtained in Ref. [1] to calculate the exact complex compliances of a CP xFRP laminate.

Five independent elasticity compliances of a UD xFRP composite have been expressed in Ref. [1] in terms of the elasticity shear compliance of a viscoelastic isotropic polymer matrix, $S_b = 1/2G$ with a shear modulus $G = E/[2(1+\nu)]$, i.e., $S_{ij}(S_b)$, $ij = 11, 22, 12, 23, 55$. The elastic-viscoelastic analogy principle gives complex compliances of a UD xFRP composite $S^*_{ij}(p) = S'_{ij}(p) + \mathrm{i}S''_{ij}(p) = S_{ij}[S^*_b(p)]$, $ij = 11, 22, 12, 23, 55$. Symbol p denotes circular frequency, $\mathrm{i} = \sqrt{-1}$, whereas $S^*_b(p)$ is a complex shear compliance of the matrix, determined in Ref. [1].

Taking into consideration Eqs. (4)-(7), six independent elasticity compliances of a regular CP xFRP laminate can be expressed in terms of the elasticity compliances $S_{ij}(S_b)$, $ij = 11, 22, 12, 23, 55$, in the form

$$\begin{aligned} &S^{cp}_{11} = \frac{1}{E_x} = \frac{2}{a\left(1-b^2\right)}, \quad S^{cp}_{33} = \frac{1}{E_z} = d + \frac{c^2}{a(1+b)} \\ &S^{cp}_{12} = -\frac{\nu_{yx}}{E_x} = -bS^{cp}_{11}, \quad S^{cp}_{13} = -\frac{\nu_{zx}}{E_x} = \tfrac{1}{2}(b-1)cS^{cp}_{11} \\ &S^{cp}_{55} = \frac{1}{G_{xz}} = S_{22} - S_{23} + \tfrac{1}{2}S_{55}, \quad S^{cp}_{66} = \frac{1}{G_{xy}} = S_{55} \end{aligned} \tag{8}$$

where

$$\begin{aligned} &a = \frac{S_{11}+S_{22}}{S_{11}S_{22}-(S_{12})^2} \\ &b = -\frac{2S_{12}}{S_{11}+S_{22}} \\ &c = -\frac{S_{12}(S_{22}-S_{12}-S_{23})+S_{11}S_{23}}{S_{11}S_{22}-(S_{12})^2} \\ &d = \frac{S_{11}(S_{22})^2-2(S_{12})^2(S_{22}-S_{23})-S_{11}(S_{23})^2}{S_{11}S_{22}-(S_{12})^2} \end{aligned} \tag{9}$$

Replacing elasticity compliances S_{ij} in Eqs. (8), (9) with complex compliances

$$S^*_{ij}(p), \; ij = 11, 22, 12, 23, 55$$

results in the exact complex compliances of a regular CP xFRP laminate, i.e.

$$\left[S^{cp}_{ij}(p)\right]^* = \left[S^{cp}_{ij}(p)\right]'_e + \mathrm{i}\left[S^{cp}_{ij}(p)\right]''_e \quad ij = 11, 33, 12, 13, 55, 66 \tag{10}$$

Subscript $_e$ denotes exact quantities.

STANDARD LVC EQUATIONS OF A REGULAR CP xFRP LAMINATE

Following the considerations presented in Ref. [1], one can formulate standard LVC equations of the homogenized regular CP xFRP laminate, satisfying the assumptions adopted in this study, in the following form

$$\boldsymbol{\varepsilon}(t) = \tilde{\mathbf{S}}(t) \otimes \boldsymbol{\sigma}(t) \tag{11}$$

where $\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(t)$ are stress and strain vectors vs. time t and

$$\tilde{\mathbf{S}}(t)=\begin{bmatrix} \tilde{S}_{11}^{cp}(t) & \tilde{S}_{12}^{cp}(t) & \tilde{S}_{13}^{cp}(t) & 0 & 0 & 0 \\ \tilde{S}_{12}^{cp}(t) & \tilde{S}_{11}^{cp}(t) & \tilde{S}_{13}^{cp}(t) & 0 & 0 & 0 \\ \tilde{S}_{13}^{cp}(t) & \tilde{S}_{13}^{cp}(t) & \tilde{S}_{33}^{cp}(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{S}_{55}^{cp}(t) & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{S}_{55}^{cp}(t) & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{S}_{66}^{cp}(t) \end{bmatrix} \tag{12}$$

is a viscoelasticity compliance matrix containing six independent viscoelasticity compliances

$$\tilde{S}_{ij}^{cp}(t)=S_{ij}^{cp}\cdot \left\{1+\alpha_{ij}\int_0^t \left[\omega_0 F_0(\vartheta;\tau_0)+\omega_1 F(\vartheta;\tau_1)+\omega_2 F(\vartheta;\tau_2)\right]d\vartheta\right\}$$
$$ij=11,33,12,13,55,66 \tag{13}$$

with

$$F_0(t;\tau_0)=\frac{1}{\tau_0}\int_0^\infty \exp\left(-\frac{ut}{\tau_0}\right)u\Lambda(u)\,du\ ,\quad \Lambda(u)=\frac{1}{\pi\sqrt{u}\,(1+u)}$$
$$F(t;\tau_i)=\frac{1}{\tau_i}\exp\left(-\frac{t}{\tau_i}\right),\quad i=1,\ 2 \tag{14}$$

The symbols used in Eqs. (11)-(14) denote:

- $\otimes$ - a convolution operator,
- $F_0(t;\tau_0)$ - a fractional exponential function $\Phi_{0.5}(t;\tau_0)$ [6],
- $F(t;\tau_i)$ - a normal exponential function $\Phi_1(t;\tau_i)$,
- $\omega_0,\omega_1,\omega_2$ - long-term creep coefficients of the matrix [3],
- τ_0,τ_1,τ_2 - retardation times ($\tau_0 << \tau_1 << \tau_2$) of the matrix [3],
- S_{ij}^{cp}, $ij=11,33,12,13,55,66$ - elasticity compliances of the CP xFRP laminate, defined by Eqs. (4),
- α_{ij}, $ij=11,33,12,13,55,66$ - relative long-term creep coefficients (RLTC) dependent of micro- and meso-structure of the laminate as well as of the viscoelastic properties of the matrix.

An analytical method for calculating RLTC coefficients for a UD xFRP composite has been presented in Ref. [1]. The method adopts the exact and approximate complex compliances of the composite and the elastic-viscoelastic analogy principle. In this study, the method developed in Ref. [1] will be extended onto regular CP xFRP laminates. Based on Eqs. (13), (14) the approximate (predicted) complex compliances of the laminate are described by the formulae

$$S_{ij}^{cp}(p)^{*}=S_{ij}^{cp}(p)'+\mathrm{i}\,S_{ij}^{cp}(p)'' ,\quad ij=11,33,12,13,55,66$$
$$S_{ij}^{cp}(p)'=S_{ij}^{cp}\left\{1+\alpha_{ij}\left[\omega_0 f_0(p\tau_0)+\omega_1 f(p\tau_1)+\omega_2 f(p\tau_2)\right]\right\}$$
$$S_{ij}^{cp}(p)''=-S_{ij}^{cp}\alpha_{ij}\left[\omega_0 g_0(p\tau_0)+\omega_1 g(p\tau_1)+\omega_2 g(p\tau_2)\right] \tag{15}$$

where [6]

$$f_0(u)=\frac{\sqrt{2}+\sqrt{u}}{\sqrt{2}+2\sqrt{u}+\sqrt{2u}}\ ,\quad g_0(u)=\frac{\sqrt{u}}{\sqrt{2}+2\sqrt{u}+\sqrt{2u}}$$
$$f(u)=\frac{1}{1+u^2}\ ,\quad g(u)=\frac{u}{1+u^2} \tag{16}$$

The RLTC coefficients, α_{ij}, are derived from the compatibility conditions put on the storage compliances at $p=0$. The final analytical formulae have the form

$$\alpha_{ij}=\frac{1}{\omega}\left[S_{ij}^{cp}(0)'_{e}\,/\,S_{ij}^{cp}-1\right],\quad ij=11,33,12,13,55,66 \tag{17}$$

where

$$\omega=\omega_0+\omega_1+\omega_2 \tag{18}$$

is termed as the total long-term creep coefficient for the matrix.

The errors of fit of the predicted complex compliances to the exact complex compliances constitute the measure of accuracy of viscoelastic modelling. The relative errors are defined by the formulae

$$\delta_{ij}'=\frac{\sum_{k=1}^{n}\left|S_{ij}^{cp}(p_k)'-S_{ij}^{cp}(p_k)'_{e}\right|}{\sum_{k=1}^{n}S_{ij}^{cp}(p_k)'_{e}}$$
$$\delta_{ij}''=\frac{\sum_{k=1}^{n}\left|S_{ij}^{cp}(p_k)''-S_{ij}^{cp}(p_k)''_{e}\right|}{\sum_{k=1}^{n}S_{ij}^{cp}(p_k)''_{e}} \tag{19}$$
$$ij=11,33,12,13,55,66$$

where p_k, $k=1,2,...,n$, are collocation points selected uniformly in the interval $p\in[p_{min},p_{max}]$ in a logarithmic scale of frequency p.

CALCULATION OF THE RLTC COEFFICIENTS FOR THE SPECIFIED LAMINATE

A regular CP CFRP laminate of $[0/90]_{nS}$, $n\geq 4$ configuration, denoted with the symbol CP U/E53, will be examined. Each ply is the UD U/E53 composite, previously considered in Refs. [1, 2]. The matrix (E53 hardening plastic) is made of Epidian 53 epoxide resin, reinforced with UTS 5631 carbon fibres produced by Tenax Fibers. The elasticity constants and the standard viscoelasticity constants of the matrix equal

$$E = 3.1\ \text{GPa}, \quad \nu = 0.42$$
$$\omega_0 = 0.23, \ \omega_1 = 0.36, \ \omega_2 = 0.11$$
$$\tau_0 = 360', \ \tau_1 = 13000', \ \tau_2 = 310000'$$

The elasticity constants of monotropic UTS 5631 carbon fibres equal

$$\overline{E}_1 = 234\ \text{GPa}, \ \overline{E}_2 = 6.6\ \text{GPa}$$
$$\overline{\nu}_{32} = 0.36, \ \overline{\nu}_{21} = 0.11, \ \overline{G}_{12} = 10.6\ \text{GPa}$$

and the fibre volume fraction $f = 0.50$.

A computer-aided algorithm presented in this study for calculation RLTC coefficients has been programmed in Pascal. For the CP U/E53 laminate one obtains the EECs and the RLTC coefficients of the following values (written here with technical accuracy):

$$E_x = E_y = 62.2\ \text{GPa}, \ E_z = 7.3\ \text{GPa}$$
$$\nu_{zx} = \nu_{zy} = 0.4, \ \nu_{yx} = \nu_{xy} = 0.02$$
$$G_{xz} = G_{yz} = 2.1\ \text{GPa}, \ G_{xy} = 2.6\ \text{GPa}$$
$$\alpha_{11} = 0.02, \ \alpha_{33} = 0.12, \ \alpha_{12} = -0.20$$
$$\alpha_{13} = 0.24, \ \alpha_{55} = 0.61, \ \alpha_{66} = 0.79$$

Diagrams presenting the selected storage and loss compliances of the laminate vs. circular frequency p in a semi-logarithmic scale are shown in Figure 1. The numerical tests have pointed out that values of the lower and upper limits must be equal to $p_{\min} = 10^{-7}$ rad/', $p_{\max} = 10^{1}$ rad/'. The collocation points p_k, $k = 1,2,...,n$, required to determine the fit errors according to Eqs. (19) have been selected uniformly in the interval $p \in [p_{\min}, p_{\max}]$ $(n = 80)$. The fit errors equal:

$$\delta'_{11} = 0.05\%, \quad \delta'_{33} = 0.22\%, \quad \delta'_{12} = 0.05\%$$
$$\delta'_{13} = 0.42\%, \quad \delta'_{55} = 0.03\%, \quad \delta'_{66} = 0.01\%$$
$$\delta''_{11} = 0.02\%, \quad \delta''_{33} = 0.09\%, \quad \delta''_{12} = 0.04\%$$
$$\delta''_{13} = 0.18\%, \quad \delta''_{55} = 0.01\%, \quad \delta''_{66} = 0.01\%$$

These values confirm excellent prediction of the viscoelastic properties of the CP U/E53 laminate. Owing to very small fit errors, the diagrams in Figure 1 reflect both the exact and predicted compliances.

CONCLUSIONS

An effective approach to viscoelastic modelling of regular cross-ply laminates has been presented. A polymer matrix of the laminate is an advanced viscoelastic isotropic material, and fibres are made of a monotropic elastic material. There are introduced the RLTC (relative long-term creep) coefficients fully describing the standard constitutive equations of linear viscoelasticity of CP xFRP laminates. A number of viscoelasticity coefficients is relatively small. A set of RLTC coefficients is calculated fully analytically via employing the exact homogenization theory of the laminate and the elastic-viscoelastic analogy principle.

A computer-aided algorithm has been used to calculate the RLTC coefficients of the specified CP CFRP laminate. Correctness, high accuracy and practical usability of the algorithm have been fully confirmed.

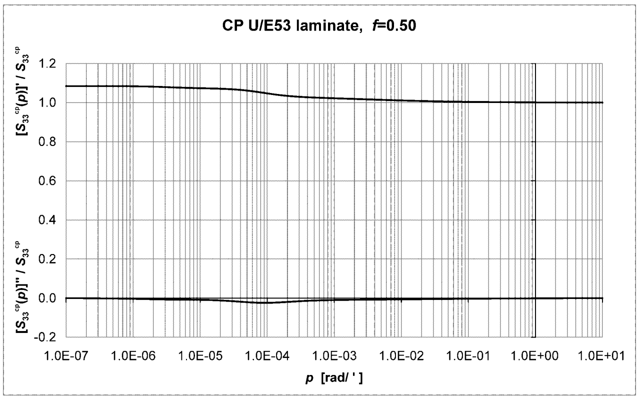

Fig. 1. The relative storage and loss compliances of the CP U/E53 laminate for $ij = 33$

Rys. 1. Względne podatności zespolone (część rzeczywista i urojona) laminatu CP U/E53 dla $ij = 33$

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