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**Maik Gude, Werner Hufenbach, Christian Kirvel***

*Institut für Leichtbau und Kunststofftechnik (ILK), TU Dresden, 01062 Dresden, Germany*
** Corresponding author. E-mail: christian.krivel@ilk.mw.tu-dresden.de*

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# STRUCTURAL ANALYSIS OF ADAPTIVE BISTABLE COMPOSITES

**Anisotropic composites are generally characterised by residual stresses, which primarily occur during the manufacturing at elevated temperature and a subsequent cooling. These residual stresses often lead to a warpage of the composite and in special cases of unsymmetrically built up composites even to large out-of-plane deformations with two stable deformation states. Those so-called bistable composites can beneficially be used for the development of novel adaptive structures if appropriate actuators are integrated, which permit to initiate a permanent change of the composite shape by a snap-through, generated only by a brief energy impulse. For the structural analysis of such adaptive bistable composites based on fibre and textile reinforced thermoplastic or thermosetting matrix systems respectively, a nonlinear semi-analytical simulation model has been elaborated, using a three-stage Ritz method according to the manufacture process of laminate consolidation, actuator bonding and actuator activation. The semi-analytical simulation model enables fast structural analysis of elementary bistable composites and efficient parameter studies. Additionally, numerical simulation models using the finite element analysis have been developed in order to perform a coupled electromechanical structural analysis of the actuator initiated snap-through behaviour of unsymmetric composites with a rather complex structural lay-up and geometry. The simulation models have been validated and were used to design first prototypes of adaptive bistable composites. Manufacture studies and experimental investigation on those prototypes illustrate the successful functional proof of novel intelligent structures and confirm the elaborated theoretical design process.**

**Keywords: unsymmetric, composites, bistable, snap-through, piezoceramic actuator**

## ANALIZA STRUKTURALNA KOMPOZYTÓW BISTABILNYCH

**Kompozyty anizotropowe są zazwyczaj charakteryzowane przez naprężenia szczątkowe, które występują głównie podczas produkcji w podwyższonej temperaturze oraz w czasie poźniejszego chłodzenia. Naprężenia te często prowadzą do wypaczenia kompozytu, a w przypadku niesymetrycznej budowy kompozytu nawet do dużych deformacji powierzchni, z dwoma stanami stabilnymi. Te tak zwane bistabilne materiały kompozytowe mogą z powodzeniem zostać zastosowane w wytwarzaniu nowoczesnych struktur adapcyjnych, jeśli odpowiednio zintegruje się elementy aktywujące, które dzięki wprowadzeniu energii w postaci impulsu inicjują przeskok (snap-through), a co się z tym wiąże - zmianę kształtu. Do analizy strukturalnej bistabilnych kompozytów na bazie tworzyw termoplastycznych lub termoutwardzalnych wzmocnionych włóknem lub tkaniną opracowano półanalityczny model obliczeniowy oraz model symulacyjny z wykorzystaniem trójpoziomowej metody Ritza, odpowiadającej procesowi wytwarzania, czyli kolejno: zagęszczanie laminatu, dołączenie czynnika aktywującego oraz jego aktywowanie. Ten półanalityczny model umożliwia szybką analizę strukturalną bistabilnych materiałów kompozytowych oraz również efektywną ocenę ich parametryzacji. Dodatkowo zostały opracowane modele numeryczne przy wykorzystaniu metody elementów skończonych w celu przeprowadzenia elektromechanicznej analizy strukturalnej elementów zjawiska przebicia inicjowanego czynnikiem aktywującym o złożonej strukturze warstwowej i złożonej geometrii. Modele te zostały zweryfikowane eksperymentalnie i z powodzeniem zastosowane do wykonania pierwszych prototypów kompozytów bistabilnych. Badania eksperymentalne oraz doświadczenia z procesu wytwarzania tych prototypów są dowodem praktycznego zastosowania nowych struktur inteligentnych i potwierdzeniem opracowanego teoretycznie procesu konstrukcyjnego.**

**Słowa kluczowe: kompozyty, niesymetryczne, dwustabilne, elementy aktywujące, przeskok**

## INTRODUCTION

Fibre and textile reinforced composites develop extensive residual stresses due to the different directional expansion of each layer during their consolidation. Those residual stresses lead to considerable out-of-plane deformations with mono- or multistable deformation states in case of unsymmetric reinforcement architectures. Initially flat formed laminates with a harmonised unsymmetric [0/90]- or[$+\theta/\theta$]-lay-up, for instance, exhibit a bistable deformation behaviour with two cylindrical distortion shapes.

These deformation states can be toggled by a snap-through of the laminate structure, which can be initiated by piezoceramic actuators [1, 2]. This actuator-controlled multiple shape characteristic can purposefully be used in the development of novel adaptive structures, in which the actuator forces and moments are only necessary to change the structures configuration but not to support the particular structure's shape [1-6]. This means an important advantage compared to conventional adaptive structures which depend on a continuous supply of power to the piezoceramic or shape memory alloy actuators in order to deform the structure elastically from its natural state of equilibrium to a "near-by" configuration.

Novel fundamental analytical and numerical dimensioning tools have been elaborated and applied, aiding to develop adaptive lightweight structures based on bistable composites. The analytical model consists of the formulation of the elastic total potential energy of the system "laminate-actuator" in combination with both the Kármán approximation to the geometrically non-linear strain-displacement relations and the application of the Rayleigh-Ritz method to determine the stationary values of the total potential energy. The developed numerical model is based on the finite element analysis (FEA) using ANSYS, which provides coupled-field brick elements designed for simulating piezoelectric properties in electro-mechanical structural analyses. The simulation models were successively used for the design of first prototypes of novel adaptive structures.

## BISTABLE DEFORMATION BEHAVIOUR OF UNSYMMETRIC COMPOSITES

The deformation behaviour of unsymmetric composites due to residual stresses is often characterised by bistable states of equilibrium [1, 2, 7-9]. These equilibrium states and the associated laminate curvatures need not be on a par as it can be seen from Fig. 1 where different states of stability due to residual stresses are exemplified by a quadratic cross-ply laminate $\left[0°_{h_1}/90°_{h_2}\right]$ made of CFRP-T3.6, with a size of 300×300×1 mm.

While the total laminate thickness h is constant, the thickness ratio $h_1/h$ of the layers is varied. Starting from a purely 0° unidirectional reinforced laminate (ratio $h_1/h$=0), which is plane, firstly a monostable cylindrical shape occurs as soon as the thickness of the 90° layer increases (Fig. 1, continuous lines). Further increasing the 90° layer thickness leads to a higher curvature $-\kappa_y$ until a maximum value is achieved for a ratio of about $h_1/h$ = 0.2. For small ratios, also a second solution is observed that is the instable state of the corresponding cylindrical shape 2 and therefore marked with a dashed line. The associated instable equilibrium state turns into a stable equilibrium state for a thickness ratio $h_1/h$ = 0.3. For a ratio between 0.3 and 0.7, two stable cylindrical shapes with generally different curvatures ($\kappa_x$ and $-\kappa_y$) exist. Only for a ratio of 0.5, these shapes are even with $\kappa_x = -\kappa_y$ otherwise either of the two cylindrical shapes preferably appears. The yellow attributed area of bistable cylindrical shapes in Fig. 1 is of high interest for the design of novel adaptive structures based on bistable composites with piezoceramic actuators.

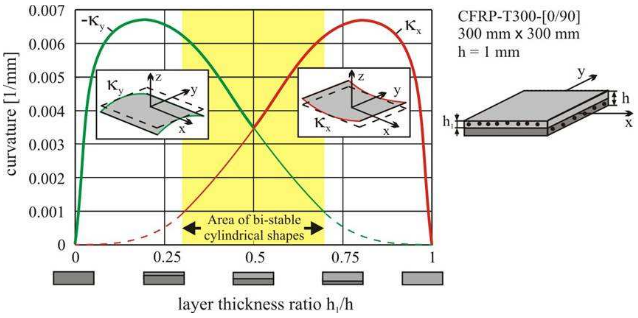


Fig. 1. Curvature and area of bistable deformation states of CFRP laminates vs. layer ratio

Rys. 1. Zakrzywienie i obszar dwustabilnych stanów odkształcenia laminatów CFRP w zależności od współczynnika grubości warstw

## SEMI-ANALYTICAL AND NUMERICAL MODELLING OF ADAPTIVE BISTABLE COMPOSITES

To take into account the large deformations - a characteristic of bistable composites - in the following semi-analytical analysis, based on the total Langrangian formulation, the linear strain-displacement relations must be extended by non-linear terms [1, 7, 10, 11]:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^0\left(\mathbf{u}^0\right) + z\boldsymbol{\kappa}\left(\mathbf{u}^0\right) = \begin{pmatrix} \dfrac{\partial u^0}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial w^0}{\partial x}\right)^2 \\ \dfrac{\partial v^0}{\partial y} + \dfrac{1}{2}\left(\dfrac{\partial w^0}{\partial y}\right)^2 \\ \dfrac{\partial u^0}{\partial y} + \dfrac{\partial v^0}{\partial x} + \dfrac{\partial w^0}{\partial x}\dfrac{\partial w^0}{\partial y} \end{pmatrix} + z\begin{pmatrix} -\dfrac{\partial^2 w^0}{\partial x^2} \\ -\dfrac{\partial^2 w^0}{\partial y^2} \\ -2\dfrac{\partial^2 w^0}{\partial x \partial y} \end{pmatrix} \tag{1}$$

where $\boldsymbol{\varepsilon}^0\left(\mathbf{u}^0\right)$ and $\boldsymbol{\kappa}\left(\mathbf{u}^0\right)$ are the strains and the curvatures of the laminate midplane respectively. The displacement field is given by $\mathbf{u}^0 = \left(u^0\, v^0\, w\right)^T$.

Analogous to the manufacture process of adaptive bistable composites, which consists of manufacture and cooling the unsymmetric laminate, bonding the actuator to the laminate and activating the system laminate-actuator by applying a voltage to the piezoceramic actuator (see Fig. 2), the developed semi-analytical model is divided into three steps. For each step, the deformation behaviour is calculated with the help of the total potential energy.

The total potential energy $\Pi_{(1)}$ of the cooled unsymmetric laminate (step 1) is given by the strain energy $U_{(1)}^{L}$ of the laminate and the potential energy of external loads $W_{th}^{L}$ according to

$$\Pi_{(1)}=U_{(1)}^{L}-W_{th}^{L}=\frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{L_y}{2}}^{\frac{L_y}{2}}\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}}\left(\boldsymbol{\varepsilon}_{(1)}^{L}\right)^{T}\mathbf{Q}^{L}\,\boldsymbol{\varepsilon}_{(1)}^{L}\,dx\,dy\,dz+$$
$$-\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{L_y}{2}}^{\frac{L_y}{2}}\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}}\left(\boldsymbol{\varepsilon}_{th}^{L}\right)^{T}\mathbf{Q}^{L}\,\boldsymbol{\varepsilon}_{(1)}^{L}\,dx\,dy\,dz \tag{2}$$

with

$$\boldsymbol{\varepsilon}_{(1)}^{L}=\boldsymbol{\varepsilon}^{0}\left(\mathbf{u}_{(1)}^{0}\right)+z\boldsymbol{\kappa}^{0}\left(\mathbf{u}_{(1)}^{0}\right) \text{ and } \boldsymbol{\varepsilon}_{th}^{L}=\boldsymbol{\alpha}^{L}\Delta T \tag{3}$$

the reduced stiffness matrix $\boldsymbol{Q}^{L}$, the coefficients of thermal expansion $\boldsymbol{\alpha}^{L}=\left(\alpha_{x}^{L},\alpha_{y}^{L},\alpha_{xy}^{L}\right)^{T}$, the side lengths of the laminate $L_x$ and $L_y$, the laminate thickness $h$, the differences in temperature with respect to the reference state $\Delta T$ and the displacement $\boldsymbol{u}_{(1)}^{0}$ after step 1. The Rayleigh-Ritz method is used to obtain approximate solutions for the displacement field $\boldsymbol{u}_{(1)}^{0}$. Therefore, general approaches for the displacement functions $u^0$, $v^0$, $w^0$ in the form of polynomials are used and the first variation $\delta\Pi_{(1)}$ is required to vanish, which results into a non-linear equation system for the unknown Ritz coefficients. The received solutions have to be checked for their stability by means of the second variation $\delta^2\Pi_{(1)}$, which has to be positive for a stable deformation state (see also [1, 2]).

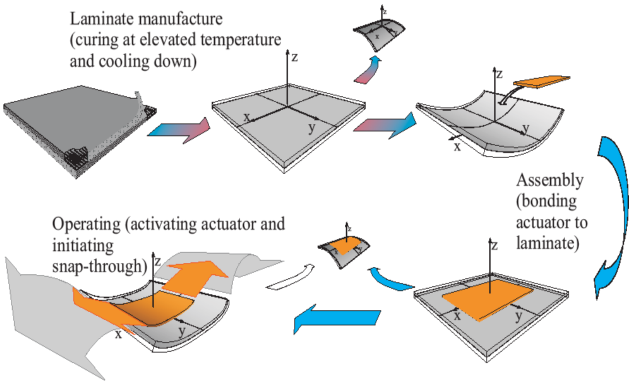


Fig. 2. Schematic diagram of the stepwise manufacturing process of an active laminate reflecting the mathematical model

Rys. 2. Schemat procesu wytwarzania aktywnego laminatu odzwierciedlający model matematyczny

Within step 2, the (passive) actuator is bonded to the laminate and the deformation of the system laminate-actuator decreases. The energetic state of the system laminate-actuator can be described by the total potential energy

$$\Pi_{(2)}=U_{(2)}^{L}+U_{(2)}^{A}=\frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{L_y}{2}}^{\frac{L_y}{2}}\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}}\left(\boldsymbol{\varepsilon}_{(2)}^{L}\right)^{T}\mathbf{Q}^{L}\,\boldsymbol{\varepsilon}_{(2)}^{L}\,dx\,dy\,dz+$$
$$+\frac{1}{2}\int_{\frac{h}{2}}^{\frac{h}{2}+h^{A}}\int_{-\frac{L_y^A}{2}}^{\frac{L_y^A}{2}}\int_{-\frac{L_x^A}{2}}^{\frac{L_x^A}{2}}\left(\boldsymbol{\varepsilon}_{(2)}^{A}\right)^{T}\mathbf{Q}^{A}\,\boldsymbol{\varepsilon}_{(2)}^{A}\,dx\,dy\,dz \tag{4}$$

with the strain energy of the laminate $U_{(2)}^{L}$, the strain energy of the actuator $U_{(2)}^{A}$, the reduced stiffness matrix of the actuator $\boldsymbol{Q}^{A}$, the side lengths of the actuator $L_x^A$ and $L_x^A$ and the actuator thickness $h^A$. The strains of the laminate in the second step based on the initial displacement $\boldsymbol{u}_{(1)}^{0}$ of step 1 are given by

$$\boldsymbol{\varepsilon}_{(2)}^{L}=\boldsymbol{\varepsilon}^{0}\left(\mathbf{u}_{(1)}^{0}+\mathbf{u}_{(2+)}^{0}\right)+z\boldsymbol{\kappa}^{0}\left(\mathbf{u}_{(1)}^{0}+\mathbf{u}_{(2+)}^{0}\right) \tag{5}$$

and the strains in the actuator by

$$\boldsymbol{\varepsilon}_{(2)}^{A}=z\boldsymbol{\kappa}^{0}\left(\mathbf{u}_{(1)}^{0}+\mathbf{u}_{(2+)}^{0}\right) \tag{6}$$

The additional displacement $\mathbf{u}_{(2+)}^{0}$ of step 2 is again calculated with the Rayleigh-Ritz method according to step 1. The total displacement after step 2 is defined as $\mathbf{u}_{(2)}^{0}:=\mathbf{u}_{(1)}^{0}+\mathbf{u}_{(2+)}^{0}$.

Within the third step, the actuator is activated by the application of an electrical voltage, which leads to a deformation of the laminate and finally to the snap-through of the laminate into the second stable deformation state. The associated total potential energy of the active system laminate-actuator can be calculated by

$$\Pi_{(3)}=U_{(3)}^{L}+U_{(3)}^{A}-W_{el}^{A}=\frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{L_y}{2}}^{\frac{L_y}{2}}\int_{-\frac{L_x}{2}}^{\frac{L_x}{2}}\left(\boldsymbol{\varepsilon}_{(3)}^{L}\right)^{T}\mathbf{Q}^{L}\,\boldsymbol{\varepsilon}_{(3)}^{L}\,dx\,dy\,dz+$$
$$+\frac{1}{2}\int_{\frac{h}{2}}^{\frac{h}{2}+h^{A}}\int_{-\frac{L_y^A}{2}}^{\frac{L_y^A}{2}}\int_{-\frac{L_x^A}{2}}^{\frac{L_x^A}{2}}\left(\boldsymbol{\varepsilon}_{(3)}^{A}\right)^{T}\mathbf{Q}^{A}\,\boldsymbol{\varepsilon}_{(3)}^{A}\,dx\,dy\,dz+$$
$$-\int_{\frac{h}{2}}^{\frac{h}{2}+h^{A}}\int_{-\frac{L_y^A}{2}}^{\frac{L_y^A}{2}}\int_{-\frac{L_x^A}{2}}^{\frac{L_x^A}{2}}\left(\boldsymbol{\varepsilon}_{el}^{A}\right)^{T}\mathbf{Q}^{A}\,\boldsymbol{\varepsilon}_{(3)}^{A}\,dx\,dy\,dz \tag{7}$$

with the strain energy of the laminate $U_{(3)}^{L}$, the strain energy of the actuator $U_{(3)}^{A}$ as well as the potential energy of external loads $W_{el}^{A}$ due to the applied voltage. Here, $\boldsymbol{\varepsilon}_{el}^{A}$ describes the electrical dilatation of the active actuator, which is calculated according to

$$\boldsymbol{\varepsilon}_{el}^{A} = \mathbf{d}E = \mathbf{d}\frac{\Delta U}{\Delta x} \tag{8}$$

with the vector of piezoelectric constants $\boldsymbol{d} = (d_{11}, d_{12}, 0)^T$ and the electrical field strength $E$ which is given by the ratio of applied voltage $\Delta U$ and distance of the electrodes $\Delta x$. The strains of the laminate are given by

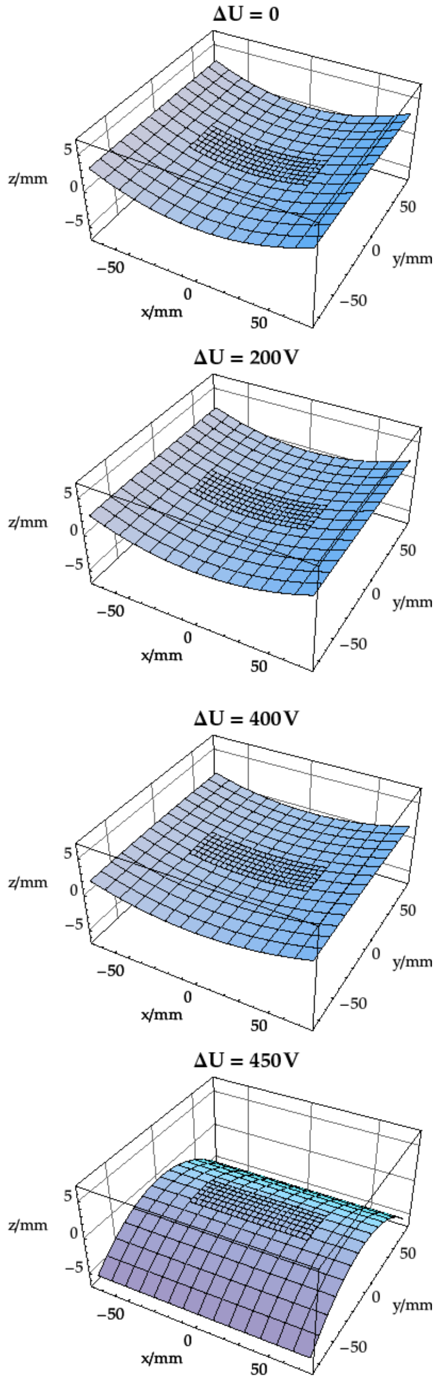
$$\boldsymbol{\varepsilon}_{(3)}^{L} = \boldsymbol{\varepsilon}^{0}\left(\mathbf{u}_{(2)}^{0} + \mathbf{u}_{(3+)}^{0}\right) + z\boldsymbol{\kappa}^{0}\left(\mathbf{u}_{(2)}^{0} + \mathbf{u}_{(3+)}^{0}\right) \tag{9}$$

whereas the strains of the active actuator can be calculated by

$$\boldsymbol{\varepsilon}_{(3)}^{A} = \boldsymbol{\varepsilon}^{0}\left(\mathbf{u}_{(3+)}^{0}\right) + z\boldsymbol{\kappa}^{0}\left(\mathbf{u}_{(2)}^{0} + \mathbf{u}_{(3+)}^{0}\right) \tag{10}$$

Based on the total potential energy $\Pi_{(3)}$, the Rayleigh-Ritz method is applied to obtain approximate solutions for the additional displacements $\mathbf{u}_{(3+)}^{0}$ of step 3. The total displacements after step 3 are then given by $\mathbf{u}_{(3)}^{0} := \mathbf{u}_{(2)}^{0} + \mathbf{u}_{(3+)}^{0}$. In an iterative process, in which the electrical voltage is successively increased, the snap-through voltage can be determined as it can be seen in the screen sequence of Figure 3 for the above mentioned model laminate.

In addition to the semi-analytical simulation model, which can efficiently be used for parameter studies on basic laminate-actuator structures, the Finite Element Analysis (FEA) has been used to develop simulation models, which enable a more advanced design of complex adaptive structures in the future. In a first step, these models have been applied to the calculation of the snap-through behaviour of basic cross-ply laminates with an embedded actuator. For the simulation, the finite-element package ANSYS has been used, which provides coupled-field brick elements (SOLID5) developed for electro-mechanical analyses with piezoelectric elements. Extensive parameter studies on the use of layered shell elements and 3D anisotropic solid elements for the unsymmetric laminate have been carried out in order to receive best numerical stability and convergence for the simulation of the snap-through effect due to voltage application. Due to compatibility and convergence reasons, the laminate itself has finally been modelled using the SOLID64 element type.


**ΔU = 0**

**ΔU = 200V**

**ΔU = 400V**

**ΔU = 450V**

Fig. 3. Sequence of the calculated deformation states of the active laminate during a snap-through simulation

Rys. 3. Kolejne etapy odkształcenia aktywnych laminatów podczas symulacji przebicia elektrycznego

The calculation of the correct deformation state due to the laminate cooling has been done by applying imperfections such as small initial, temporary forces or deformations, which initiate the correct deformation during the non-linear computation with a stepwise thermal loading of $\Delta T = -100$ K in total (see also [1]). After the last thermal load step, an electrical field in the actuator is gradually applied in order to expand the piezoceramic actuator and to finally initiate the snap-through of the laminate-actuator structure.

## EXPERIMENTAL VERIFICATION

Based on the theoretical investigations, first prototypes of active bistable composites have been manufactured and successfully tested. Among others, a CFRP-T3.6-laminate (150×150× 0.5 mm) has been cured at 120°C and a Macro-Fibre Composite (MFC, by Smart Material GmbH) type M8528 has been bonded on one side of the laminate. Fig. 4 shows the comparison of the deformation geometries that were achieved by the two theoretical models and by the aid of an optical measurement system on that specimen.

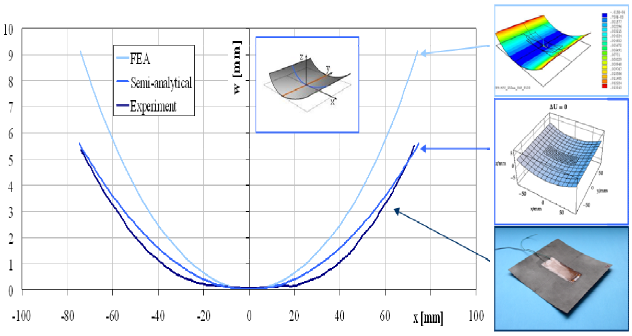


Fig. 4. Experimentally and theoretically determined deformations (sections along x-axis) of a cross-ply CFRP-T3.6-laminate (150x150x0.5 mm³) equipped with a piezoelectric actuator system (MFC) in passive condition at room temperature

Rys. 4. Eksperymentalnie i teoretycznie wyznaczone odkształcenia (przekroje wzdłuż osi x) w laminacie CFRP-T3.6 (150x150x0,5 mm³) wyposażonym w aktywator piezoelektryczny (MFC) w warunkach normalnych

Finally, an electrical voltage has been supplied to the MFC in order to initiate the snap through of the laminate-actuator system into the second deformation state. The average snap-through voltage of several tests was found to be 536 V.

Analytical and numerical simulations as much as the experimental tests show a qualitative good agreement in the actuator initiated structural behaviour of laminate-actuator systems, however the quantitative comparison in the experimentally and theoretically determined snap-through voltage shows slight differences. It is felt that the influence of the bonding layer between laminate and MFC to the actuation transfer from actuator to laminate and thus to the deformation behaviour cannot be neglected as it has been done so far. Besides that, the theoretical models will be refined by the use of a higher element number for the FEA or by higher order polynomials for the displacement approaches for the semi-analytical model respectively.

## CONCLUSIONS

The purposeful use of residual stresses in composite structures in combination with actuators enables the development of novel adaptive lightweight structures. For the structural analysis of active bistable composites, a novel semi-analytical, geometrically non-linear simulation model using the Rayleigh-Ritz technique has been elaborated. The model enables the electromechanical structural analysis and the iterative determination of the electrical voltage to initiate the snap-through of basic bistable laminate-actuator structures.

Additionally, the capability to simulate the snap-through behaviour with the help of FEA using ANSYS has successfully been investigated. With the help of coupled-field brick elements for modelling the piezoceramic actuator, the numerical simulation of the snap-through is possible. The numerical model also offers the possibility to simulate more complex bistable structures in the future.

On the basis of the theoretical investigations, first prototypic adaptive bistable composite structures have been manufactured and a functional proof has successfully been performed. Future work will also include further experimental investigations of bistable composites as well as of more complex morphing structures.

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