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*Received (Otrzymano) 13.01.2022*

## MODELLING OF LINEAR ELASTICITY AND VISCOELASTICITY OF THERMOSETS AND UNIDIRECTIONAL GLASS FIBRE-REINFORCED THERMOSET-MATRIX COMPOSITES – PART 1: THEORY OF MODELLING

The paper presents advanced analytical modelling of the linear elasticity and viscoelasticity of thermosets and unidirectional long glass fibre-reinforced thermoset-matrix (UFRT) composites. New non-aging materials fully relaxed after the curing and post-curing processes are considered. Quasi-static long-term isothermal reversible viscoelastic processes under normal conditions are modelled. The thermosets are isotropic materials with viscoelastic shear strains and elastic bulk strains, and the fibres are isotropic and elastic. New rheological models for thermosets and UFRT composites, described by the smallest possible number of material constants, are developed. The viscoelastic generic function for shear/quasi-shear stresses is assumed as the Mittag-Leffler fractional exponential function in an integral form. The thermoset is described by two elastic and three viscoelastic parameters. The homogenized UFRT composite is described by five elastic and five viscoelastic parameters. Conjugated/unconjugated standard/inverse constitutive equations of the linear elasticity/elasticity-viscoelasticity governing thermosets and UFRT composites are formulated. The equations are mutually analytically transformable.

Keywords: thermoset, unidirectional glass fibre-reinforced thermoset-matrix composite, rheological modelling, constitutive equations of linear elasticity-viscoelasticity, analytical modelling

### INTRODUCTION

Laminates made of a thermoset matrix reinforced with stitched glass fabrics, commonly labelled GFRP (glass fibre-reinforced plastic), belong to the most widely used composite materials in civil, mechanical, automotive and naval engineering, e.g. tanks, covers, bridge platforms, footbridges, and yachts. Each lamina is a unidirectional fibre-reinforced thermoset-matrix (UFRT) composite. For modern mechanical manufacturing technologies, e.g. vacuum infusion and resin transfer moulding, the fibre volume fraction of a UFRT composite is around 50%. In order to complete the cross-linking process and achieve the maximum heat distortion temperature (HDT) of certain thermoset (T) matrices, adequate post-curing of laminate shells at elevated temperatures may be required.

In the finite element method (FEM) modelling of laminate shells, it is generally accepted that the thermoset and the glass fibre are linearly elastic and brittle isotropic materials. As a result of homogenization, each lamina is modelled as a monotropic (transversely isotropic), homogeneous, linearly elastic and brittle material, e.g. Daniel and Ishai [1]. In fact, a post-cured thermoset matrix is a semi-solid material and exhibits viscoelastic or viscoelastic-plastic properties depending on the stress, strain and temperature levels. These

properties of GFRP laminate structures may significantly affect the long-term stress redistribution or induce undesirable long-term deformations.

Publications on the quasi-static viscoelasticity and plasticity of T polymers and UFRT composites can be divided into two groups:

1. the viscoelastic response of T and UFRT during the manufacturing process (representative Refs. [2-7]);
2. the elastic-viscoelastic or elastic-viscoelastic-plastic response of T and UFRT new materials, fully relaxed after the manufacturing process, to long-term mechanical loads (representative Refs. [8-39]).

During the manufacturing process, residual stresses/strains appear in the UFRT material due to crosslink induced shrinkage. After fabrication, the material is conditioned in normal conditions and the residual stresses/strains are fully relaxed. The present work belongs to the second group of publications.

Constitutive equations of the elastic-viscoelastic/elastic-viscoelastic-plastic response of thermosets fully relaxed after the manufacturing process can be reflected by mechanical models composed of Hooke (H), Kelvin (K), Maxwell (M), Newton (N), Rabotnov (R), Schapery (S) elements and others, in a serial or parallel connection. H, K, M, and N elements are defined classically.

An R element is the parallel combination of a Hooke's element and a damping element described by the Mittag-Leffler fractional exponential generic function in an integral form (Rabotnov [10]). An S element uses a fractional power generic function (Schapery [8]).

The H-S or modified H-S model was applied to T/UFRT by Aboudi [11], Papanicolaou et al. [13], Haj-Ali and Muliana [16], Falahatgar and Salehi [19], Jeon et al. [21], and Nunes et al. [28]. Ferry [9] developed the H- $\infty$ K rheological model for polymeric materials, also used by Starkova and Aniskevich [15].

Papanicolaou et al. [13] presented new algorithms to estimate the nonlinear viscoelastic parameters relative to the modified Schapery's single constitutive equation. The validity of the viscoelastic model was verified through creep-recovery experiments on a 90° unidirectional carbon-fibre/epoxy-matrix composite. Six stress levels, from 30 to 70% of the tensile strength, were applied. The creep-recovery process lasted 14 days.

Kontou [14] studied the tensile creep of a unidirectional glass-fibre/epoxy-vinyl-resin-matrix composite at the temperatures of 25°C, 60°C, and 80°C. The testing was performed on the pure matrix and on composite specimens cut at four orientations: 15°, 30°, 60°, and 90°. The nonlinear creep strain included an elastic, a visco-elastic and a viscoplastic part. It was assumed that the accumulation of strain in a creep procedure follows a thermally activated mechanism, while the plastic part of strain could be calculated separately following the Spathis-Kontou function. The plastic deformations occur around a large number of voids or defects, randomly distributed into the volume of the material.

Starkova and Aniskevich [15] considered the limits of the linear viscoelastic behaviour of selected polymers, including thermoplastics and thermosets (epoxy, polyester, vinyl ester). The creep function is a sum of normal exponents expressed in terms of discrete retardation spectrum. The value of the stored deviatoric energy is considered as a limit of linear viscoelastic behaviour. The experimental data on tension at various constant strain rates and tensile creep at various stresses, temperatures, and in various moisture conditions were considered.

Haj-Ali and Muliana [16] introduced an integrated micromechanical-sublaminar modelling approach for the nonlinear viscoelastic analysis of thick laminates and structures. Schapery's single integral constitutive model corresponding to the uniaxial nonlinear creep strain was used. The proposed modelling approach was calibrated and verified against one-hour off-axis creep tests on the unidirectional glass/epoxy composite.

Muliana and Savant [17] formulated a concurrent micromechanical model for predicting the effective responses of unidirectional fibre-reinforced polymer-matrix composites. Different viscoelastic material properties for the orthotropic fibre and isotropic matrix were allowed to change with the temperature field. A linearized thermo-viscoelastic constitutive model for orthotropic fibre materials was formulated. This model was reduced to a linear elastic response in the case of

carbon or glass fibres. It was simplified to isotropic viscoelastic behaviour for the polymer matrix. The transient compliance in the creep function followed the Prony series exponential form.

Ascione et al. [18] considered the long-term linear viscoelastic behaviour of fibre-reinforced polymer-matrix composites. The fibre and matrix phases were described by a four-parameter rheological law.

Falahatgar and Salehi [19] examined the nonlinear bending analysis of a polymer-matrix laminated plate. The unit cell in the three-dimensional closed-form solution was used to describe the behaviour of the unidirectional composite. The elastic fibres were transversely isotropic and Schapery's single integral constitutive equation in a multiaxial stress state described the isotropic polymeric matrix. Four nonlinear viscoelastic parameters were expressed in terms of the effective octahedral shear stress. The deflections of the Mindlin annular sector graphite/epoxy plate under uniform lateral pressure with clamped and hinged edge constraints were simulated and compared with the elastic finite element results.

The prediction of the creep failure times for selected polymers and one polymer-matrix composite was analysed by Spathis and Kontou [20]. The following materials were tested experimentally: Nylon 66 (thermoplastic), a glass-fibre nylon-matrix composite, PC Lexan 141 (thermoplastic), and T300/VE (a carbon-fibre vinyl-ester-matrix composite). The creep failure times, defined as the time when the creep rate reaches the minimum, were determined for the above specified materials at a variety of stress levels. Creep rupture at high levels of stress is caused by large viscoelastic strains, primary and secondary bond rupture, shear yielding and cracking, chain slippage, void formation and growth, as well as fibre breakdown. The researchers concluded that temperature, the stress level and time are crucial factors for the creep response of plastics and FRP composites.

Jeon et al. [21] presented two micromechanical models for analysing a time-dependent inelastic response of unidirectional fibre-reinforced polymer-matrix composites. The fibres were considered as a linearly elastic monotropic medium, while the polymeric matrix was a viscoelastic-viscoplastic isotropic medium. Schapery's nonlinear viscoelastic model was combined with the Perzyna viscoplastic model and used for the polymeric matrix. All the components of the deviatoric and volumetric stresses and strains were considered as time-dependent and a tensor calculus was applied. The first model was based on a simplified fibre-matrix unit-cell model. The second model considered the effects of the fibre arrangements, stress concentrations, and stress discontinuities on the overall response of the composite.

Müller et al. [22] presented the finite element implementation of linear and nonlinear fractional viscoelasticity Maxwell models. The analytical and numerical solution schemes for selected fractional integral and differential equations were elaborated. The uniaxial constitutive relations of linear and nonlinear fractional

viscoelasticity were formulated, generalized to the multiaxial case and processed to be implemented in an FE code.

Zhang et al. [23] developed a new method for predicting the viscoelastic behaviour of unidirectional-fibre-reinforced polymer-matrix composites. The H-K viscoelastic model of the matrix was used to build two simplified uncoupled viscoelastic models for the composite tensioned in  $0^\circ$  and  $90^\circ$  directions relative to the fibre direction. The fibre model was linear elastic. The viscoelastic parameters were identified based on modified 4.5-hour creep tests.

Kotelnikova-Weiler et al. [24] investigated, both experimentally and numerically, the creep rupture phenomenon of a unidirectional-glass-fibre-reinforced vinylester-matrix composite rod subjected to a constant combined shear-traction load. A power law for the creep function of the matrix was assumed. The experiments were performed at different high levels of load, at temperatures of 20 and  $60^\circ\text{C}$ .

Cardoso and Harries [25] formulated a linear viscoelastic model for unidirectional-glass-fibre-reinforced polymer-matrix composites. The longitudinal rovings and continuous strand mat with randomly-oriented fibres were assumed to be elastic, while the polymer matrix was described by the M-K viscoelastic model. A viscoelastic extension of the rule of mixtures was adopted and explicit equations for the creep strain and strain rate were derived. To validate this model, flexural creep (5 days) and recovery (1 day) experimental tests were carried out on pultruded vinylester and orthopolyester-based composite specimens with  $0^\circ$  or  $90^\circ$  E-glass fibre orientation.

Sejnoha et al. [27] analysed the nonlinear viscoelastic response of unidirectional basalt and carbon-fibre-reinforced polymer-matrix composites. The researchers derived the stress-strain curves as functions of the loading rate by means of experiments and macromechanical FE simulations. The macroscopic stress-strain curves provided by FEM were compared with the micromechanical Mori-Tanaka estimates. Classic creep tensile tests were conducted for 2 hours at small, medium and high levels of constant stress. The nonlinear viscoelastic Leonov model, equivalent to the generalized Maxwell chain model, for the polymer matrix was assumed.

Nunes et al. [28] compared dynamic mechanical and thermal analysis in the time-temperature superposition principle mode with one-hour macrotests at elevated temperatures, in order to determine the viscoelasticity master curves of polymers. Schapery's linear viscoelastic model containing Prony series was applied, described by a single constitutive equation for uniaxial loading. Experimental tensile macrotests were conducted on the selected epoxy system at temperatures of  $30\text{--}110^\circ\text{C}$ .

Berardi et al. [29] presented the results from 42-month creep experiments in the fibre direction on a uniaxial E-glass fibre-reinforced epoxy-matrix composite at room temperature under different stress levels, i.e. 12.2-26.5% of the tensile strength. The experimental

results were compared with the predictions by linear/nonlinear viscoelastic models taken from the literature, i.e. the Burgers, Findley, and Nutting models.

The research described in Refs. [8-29] can be summarized as follows:

- Various models are used to describe the viscoelastic and plastic behaviour of T and UFRT materials, e.g. H-S, modified H-S, fractional M, H-K, M-K, Leonov, Burgers, Findley, Nutting and others. These models are applied to different polymers or polymer-matrix composites subjected to quasi-static constant loads at different levels and temperatures.
- The viscoelastic modelling of UFRT composites is most often based directly on experimental creep or creep-recovery tests. It is then not possible to theoretically predict the viscoelastic behaviour of composites with different components and different fibre fractions.
- A promising tool for studying the viscoelastic behaviour of thermosets is the time-temperature superposition principle. According to this principle, the response of a linearly viscoelastic thermoset to quasi-static loading over a large time span at ambient temperature is equivalent to their response over a much shorter time span at an elevated temperature.
- To date, the effect of the aging of T and UFRT materials on their long-term viscoelastic properties has not been studied.
- To date, a quasi-static linear elastic-viscoelastic H-R/H (shear/bulk) model of thermosets, with the generic function in the integral form of a Mittag-Leffler fractional exponential function, has not been formulated. This function aptly reproduces the gradients of the shear strains in the material during long-term creep induced by constant shear stresses at ambient temperature, as will be demonstrated in this paper.
- To date, no standard or inverse constitutive equations for the linear elasticity-viscoelasticity of a UFRT composite, based on a single fractional exponential generic function describing the thermoset matrix, have been formulated. It will be demonstrated in this paper.

The H-R rheological model is a modification of the H-K rheological standard model; the Kelvin element is replaced by the Rabotnov element.

Klasztorny, Wilczynski, Witember-Perzyk, Gieleta, Nycz and Bogusz [30-39] made a significant contribution to the research of the linear elastic-viscoelastic/elastic-viscoelastic-viscous modelling of thermosets and UFRT composites. The H-R/H-R (shear/bulk) rheological model for thermosets was formulated by Wilczynski and Klasztorny [30]. This model was applied to determine the complex compliances of a UFRT composite using the elastic compliances of the composite and the viscoelastic-elastic correspondence principle. Standard elastic-viscoelastic constitutive equations for a UFRT composite, corresponding to the H-R/H-R model of a thermoset matrix were formulated by Klasztorny and

Wilczynski [31]. The researchers formulated a numerical algorithm to estimate 27 viscoelastic constants describing that model.

The M-R-K rheological model for thermosets and thermoplastics, described by 14 elastic/viscoelastic/viscous constants, was formulated by Klasztorny et al. [32]. The researchers formulated a numerical algorithm to estimate the material constants and tested it on a polyester resin. The H-R-2K rheological model for thermosets, described by 9 elastic/viscoelastic constants, was formulated in Refs. [33-35]. This model adopted the half fractional exponential generic function in the integral form. Standard elastic-viscoelastic constitutive equations governing UFRT composites were formulated; the approach resulted in 18 elastic/viscoelastic constants. The numerical algorithms to estimate the material constants of the matrix and composite were formulated as well.

The H-R-2K/H (shear/bulk) rheological model for thermosets, described by a half fractional exponential generic function, two normal exponential generic functions and 8 elastic/viscoelastic constants, was formulated by Klasztorny [36, 37]. The standard/inverse unconjugated/conjugated elasticity/viscoelasticity equations of the material were formulated. The respective numerical algorithms to estimate the material constants were elaborated, programmed and tested on polyester and epoxy resins. An analytical-numerical algorithm to simulate reversible rheological processes in thermosets was developed and tested by Klasztorny [38].

The creep effect in cross-ply FRP laminates manifested significantly in the in-plane shear test (Klasztorny et al. [39]). According to the EN ISO 14129 standard, the in-plane shear modulus and in-plane shear strength for cross-ply composites are identified from a  $\pm 45^\circ$  off-axis tensile test at the crosshead displacement rate of 2 mm/min. The researchers presented the results of experimental quasi-static  $\pm 45^\circ$  off-axis tensile tests at various displacement rates, aimed at demonstrating that the high nonlinearity of shear stress-shear strain curves is caused by viscoelastic flow of the resin at low and medium levels of the shear stress and by viscoelastic-plastic flow of the resin at high levels of the shear stress.

The research described in Refs. [30-39] can be summarized as follows:

- The H-R/H-R rheological model for thermosets is inadequate as it is based on incorrect experimental data related to viscoelastic bulk deformations.
- The M-R-K, H-R-2K, H-R-2K/H rheological models for thermosets are described by relatively large numbers of viscoelastic material constants (fuzzy sets), leading to major discrepancies in the theoretical predictions from experimental results.
- To date, only standard elastic-viscoelastic constitutive equations for UFRT composites, corresponding to the H-R/H-R and H-R-2K rheological models of thermosets, have been formulated.

The present study on fully analytical modelling of the linear elasticity and viscoelasticity of thermosets and

UFRT composites has been divided into two parts. Part 1 (this manuscript) consists of the following sections: an introduction including the state of the art, research objectives, assumptions, modelling of the linear elasticity and viscoelasticity of thermosets, and modelling of the linear elasticity and viscoelasticity of UFRT composites. Part 2 (a subsequent manuscript [40]) includes the following sections: the modified homogenization of UFRT composites, a description of new numerical algorithms corresponding to the analytical solutions presented in Part 1, a description and analysis of the experimental and numerical tests on the representative materials.

Parts 1 and 2 present a new approach to fully analytical modelling of the linear elasticity and viscoelasticity of thermosets and UFRT composites. This approach is based on:

- a Mittag-Leffler fractional exponential function in an integral form as a generic function for shear stresses in a thermoset;
- unconjugated standard constitutive equations of the linear elasticity of a thermoset;
- unconjugated inverse constitutive equations of the linear elasticity of a thermoset;
- unconjugated standard constitutive equations of the linear elasticity and viscoelasticity of a thermoset;
- the complex shear compliance of a thermoset;
- unconjugated inverse constitutive equations of the linear elasticity and viscoelasticity of a thermoset;
- the complex shear stiffness of a thermoset;
- unconjugated standard constitutive equations of the linear elasticity of a homogenized UFRT composite;
- unconjugated inverse constitutive equations of the linear elasticity of a homogenized UFRT composite;
- complex quasi-shear compliances of a homogenized UFRT composite;
- complex quasi-shear stiffnesses of a homogenized UFRT composite;
- modified homogenization theory of a UFRT composite;
- n-point Gauss-Legendre quadrature;
- fitting of the simulation curve to the experimental curve of long-term shear deformation vs. log (time) in the tensile creep test of a thermoset in normal conditions and a low level of normal stress;
- the viscoelastic-elastic correspondence principle applied to a UFRT composite.

## RESEARCH OBJECTIVES AND ASSUMPTIONS

In this study, thermosets (thermosetting resins) and UFRT composites are considered. The main research objectives are as follows:

- to develop new rheological models (coded H-R/H) for thermosets and UFRT composites, described by the smallest possible number of material constants, leading to technical agreement with the long-term viscoelastic behaviour of real materials;

- to formulate constitutive equations of elasticity and viscoelasticity, standard and inverse, unconjugated and conjugated, describing thermosets and UFRT composites (all the transformations are to be fully analytical);

The following assumptions are made in modelling the elasticity and viscoelasticity of thermosets and UFRT composites, valid for structural design:

- a new non-aging material fully relaxed after the curing and post-curing processes (free of residual stresses);
- quasi-static long-term isothermal viscoelastic processes;
- normal conditions;
- low levels of stresses in the thermoset matrix providing reversibility of the elastic and viscoelastic processes;
- long continuous rectilinear fibres arranged unidirectionally and uniformly in the matrix in a hexagonal scheme;
- a cylindrical representative volume cell (RVC) equivalent in volume to the true hexagonal cell;
- a linear viscoelastic isotropic thermoset described by the H-R/H (shear/bulk) rheological model;
- linear elastic isotropic fibres with an identical circular cross section;
- a linear viscoelastic monotropic composite after homogenization.

## MODELLING OF LINEAR ELASTICITY AND VISCOELASTICITY OF UFRT COMPOSITES

The assumptions made for modelling the elasticity and viscoelasticity of thermosetting plastics are given in previous Section. The modelling was carried out in the  $x_1x_2x_3$  Cartesian coordinate system, coinciding with the directions of the monotropy of the homogenized UFRT composite. The stress and strain components in the  $x_1x_2x_3$  system are defined classically, i.e.:  $\sigma_1, \sigma_2, \sigma_3$  – normal stresses,  $\tau_{23}, \tau_{13}, \tau_{12}$  – shear stresses,  $\varepsilon_1, \varepsilon_2, \varepsilon_3$  – normal strains,  $\gamma_{23}, \gamma_{13}, \gamma_{12}$  – shear strains. The linear elasticity of a thermoset is described by two elastic constants, i.e.  $E$  – Young's modulus,  $\nu$  – Poisson's ratio. The shear and bulk modules are respectively

$$G = E/2(1 + \nu), \quad B = E/3(1 - 2\nu) \quad (1)$$

In matrix notation, well-known conjugated standard constitutive equations of the linear elasticity of a thermoset can be written as

$$\boldsymbol{\varepsilon} = \mathbf{S} \boldsymbol{\sigma}, \quad \boldsymbol{\delta} = S_s \boldsymbol{\tau} \quad (2)$$

where

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix}, \quad \boldsymbol{\delta} = \begin{bmatrix} \delta_{23} \\ \delta_{13} \\ \delta_{12} \end{bmatrix}, \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}, \quad \boldsymbol{\tau} = \begin{bmatrix} \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} S_{11} & S_{12} & S_{12} \\ S_{12} & S_{11} & S_{12} \\ S_{12} & S_{12} & S_{11} \end{bmatrix},$$

$$S_{11} = 1/E, \quad S_{12} = -\nu/E, \quad S_s = 1/2G \quad (3)$$

with  $\delta_{jk} = \gamma_{jk}/2, jk = 23, 13, 12$ . Vectors  $\boldsymbol{\varepsilon}, \boldsymbol{\delta}, \boldsymbol{\sigma}, \boldsymbol{\tau}$  represent parts of the strain and stress tensors at point  $(x_1, x_2, x_3)$ , respectively. Matrix  $\mathbf{S}$  is the elastic compliance matrix and coefficient  $S_s$  is the elastic shear compliance of a thermoset. Equations (2)<sub>2</sub> are unconjugated.

Well-known conjugated inverse constitutive equations of the linear elasticity of a thermoset can be written in the following form

$$\boldsymbol{\sigma} = \mathbf{C} \boldsymbol{\varepsilon}, \quad \boldsymbol{\tau} = C_s \boldsymbol{\delta} \quad (4)$$

where

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{12} \\ C_{12} & C_{11} & C_{12} \\ C_{12} & C_{12} & C_{11} \end{bmatrix}, \quad C_{11} = E(1 + \nu)(1 - \nu)/\Delta, \quad C_{12} = E(1 + \nu)\nu/\Delta$$

$$\Delta = 1 - 3\nu^2 - 2\nu^3, \quad C_s = 2G \quad (5)$$

Vectors  $\boldsymbol{\varepsilon}, \boldsymbol{\delta}, \boldsymbol{\sigma}, \boldsymbol{\tau}$  are defined in Eqns. (3)<sub>1-4</sub>. Matrix  $\mathbf{C}$  is the elastic stiffness matrix and coefficient  $C_s$  is the elastic shear stiffness of a thermoset.

Well-known unconjugated standard constitutive equations of the linear elasticity of a thermoset modelled as an isotropic material, equivalent to Eqn. (2)<sub>1</sub>, can be written as (Ref. [32])

$$\boldsymbol{\varepsilon}_s = S_s \boldsymbol{\sigma}_s, \quad \boldsymbol{\varepsilon}_b = S_b \boldsymbol{\sigma}_b \quad (6)$$

where

$$\boldsymbol{\varepsilon}_s = \begin{bmatrix} \varepsilon_1 - \varepsilon_b \\ \varepsilon_2 - \varepsilon_b \\ \varepsilon_3 - \varepsilon_b \end{bmatrix}, \quad \boldsymbol{\sigma}_s = \begin{bmatrix} \sigma_1 - \sigma_b \\ \sigma_2 - \sigma_b \\ \sigma_3 - \sigma_b \end{bmatrix}$$

$$\boldsymbol{\varepsilon}_b = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)/3, \quad \boldsymbol{\sigma}_b = (\sigma_1 + \sigma_2 + \sigma_3)/3, \quad S_b = 1/3B \quad (7)$$

Vectors  $\boldsymbol{\varepsilon}_s, \boldsymbol{\delta}, \boldsymbol{\sigma}_s, \boldsymbol{\tau}$  represent parts of the strain and stress deviators of an isotropic material at point  $(x_1, x_2, x_3)$ , accordingly. Quantities  $\boldsymbol{\varepsilon}_b, \boldsymbol{\sigma}_b$  represent the strain and stress axiators of an isotropic material at the same point, respectively. Coefficient  $S_b$  is the elastic bulk compliance.

Unconjugated inverse constitutive equations of the linear elasticity of a thermoset are obtained from Eqns. (6), i.e.

$$\boldsymbol{\sigma}_s = C_s \boldsymbol{\varepsilon}_s, \quad \boldsymbol{\sigma}_b = C_b \boldsymbol{\varepsilon}_b \quad (8)$$

where  $C_b = 3B$  is the elastic bulk stiffness.

Based on Eqns. (2)-(8), the relationships between the elastic-compliance/elastic-stiffness matrices and the elastic shear/bulk compliance/stiffness have the form

$$\mathbf{S} = S_s (\mathbf{I} - \mathbf{A}) + S_b \mathbf{A}, \quad \mathbf{C} = C_s (\mathbf{I} - \mathbf{A}) + C_b \mathbf{A}$$

$$\mathbf{I} = \text{diag}(1, 1, 1), \quad \mathbf{A} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad (9)$$

According to the adopted assumptions, unconjugated standard constitutive equations of the linear elasticity-viscoelasticity of a thermoset, corresponding to Eqns. (6), (2)<sub>2</sub>, have the following form:

$$\begin{aligned} \varepsilon_s(t) &= \tilde{S}_s(t) \otimes \sigma_s(t), \quad \varepsilon_b(t) = S_b \sigma_b(t) \\ \delta(t) &= \tilde{S}_s(t) \otimes \tau(t) \end{aligned} \quad (10)$$

for time variable  $t \geq 0$ , with

$$\begin{aligned} \tilde{S}_s(t) &= S_s \left[ 1 + c \int_0^t \Phi(v) dv \right], \quad \Phi(t) = \frac{1}{T_c} \int_0^\infty \exp\left(-\frac{ut}{T_c}\right) u L(u) du \\ L(u) &= \frac{\sin \pi r}{\pi u} \frac{u^r}{1 + 2u^r \cos \pi r + u^{2r}}, \quad 0 < r < 1 \end{aligned} \quad (11)$$

The nomenclature and interpretation are as follows:  $L(u)$  – function creating the fractional exponential function in an integral form,  $r$  – fraction creating the fractional exponential function,  $T_c$  – retardation time,  $\Phi(t)$  – generic function for shear stresses (the Mittag-Leffler fractional exponential function in an integral form),  $\tilde{S}_s(t)$  – elastic-viscoelastic shear compliance,  $c$  – long-term creep coefficient,  $\otimes$  – convolution product operator.

The Mittag-Leffler fractional exponential function in an integral form was formulated by Rabotnov [10] and used as a generic function for polymers. The elastic-viscoelastic model of a thermoset described by Eqns. (10, 11) will be labelled H-R/H. The mechanical representation of this model consists of two parts: a Hooke elastic element and a Rabotnov viscoelastic element connected in a series for the strain deviator and a Hooke element for the strain axiator. In the limit case  $r = 1$ , a Rabotnov element transforms into a Kelvin element (a H-R/H model transforms into a H-K/H model).

Further transformations will be performed for Eqn. (10)<sub>1</sub> because the transformations for Eqn. (10)<sub>3</sub> are analogous. The convolution product in Eqn. (10)<sub>1</sub> results in

$$\varepsilon_s(t) = S_s \left[ \sigma_s(t) + c \int_0^t \Phi(v) \sigma_s(t-v) dv \right] \quad (12)$$

The stress programme for shear creep has the form

$$\sigma_s(t) = \sigma_{s0} H(t) \quad (13)$$

where:  $\sigma_{s0}$  – initial shear stresses,  $H(t)$  – the Heaviside function. The shear strain response to the shear stress programme results from Eqn. (12) and amounts to

$$\varepsilon_s(t) = S_s [1 + c \varphi(t)] \sigma_{s0} \quad (14)$$

where

$$\varphi(t) = \int_0^t \Phi(v) dv = 1 - \int_0^\infty \exp\left(-\frac{ut}{T_c}\right) L(u) du \quad (15)$$

is a creep function.

Functions  $L(u)$ ,  $\Phi(t)$ ,  $\varphi(t)$  have the following properties:

$$\begin{aligned} L(u) &\geq 0 \text{ for } u \geq 0, \quad \int_0^\infty L(u) du = 1, \quad \lim_{u \rightarrow 0^+} L(u) = \infty \\ \Phi(t) &> 0 \text{ for } t > 0, \quad \lim_{t \rightarrow 0^+} \Phi(t) = \infty, \quad \lim_{t \rightarrow \infty} \Phi(t) = 0 \\ \varphi(0) &= 0, \quad \varphi(\infty) = \lim_{t \rightarrow \infty} \varphi(t) = 1 \end{aligned} \quad (16)$$

The initial creep velocity of a thermoset has an infinite value, i.e.

$$\dot{\varphi}(t) = \frac{d\varphi(t)}{dt} = \Phi(t), \quad \dot{\varphi}(0) = \lim_{t \rightarrow 0^+} \Phi(t) = \infty \quad (17)$$

For  $t > 0$  velocity  $\dot{\varphi}(t)$  drops sharply to a low value, as observed in the experimental tests.

The Laplace transform made on Eqn. (10)<sub>1</sub> has the form:

$$\bar{\varepsilon}_s(q) = S_s [1 + c \bar{\Phi}(q)] \bar{\sigma}_s(q) \quad (18)$$

where  $q$  is a complex variable, and

$$\begin{aligned} \bar{\varepsilon}_s(q) &= \int_0^\infty \exp(-qt) \varepsilon_s(t) dt, \quad \bar{\sigma}_s(q) = \int_0^\infty \exp(-qt) \sigma_s(t) dt \\ \bar{\Phi}(q) &= \int_0^\infty \exp(-qt) \Phi(t) dt = \frac{1}{1+(qT_c)^r} \end{aligned} \quad (19)$$

are the Laplace transforms of  $\varepsilon_s(t)$ ,  $\sigma_s(t)$ ,  $\Phi(t)$ , respectively.

For the harmonic stress programme, i.e.

$$\sigma_s^*(t) = \sigma_{s0} \exp(i\omega t) \quad (20)$$

the elastic-viscoelastic response described by Eqn. (10)<sub>1</sub> is

$$\begin{aligned} \varepsilon_s^*(t) &= \tilde{S}_s(t) \otimes \sigma_s^*(t) = S_s \left[ 1 + c \int_0^t \Phi(v) dv \right] \otimes \sigma_{s0} \exp(i\omega t) = \\ &= S_s \left\{ \exp(i\omega t) + c \int_0^t \exp[i\omega(t-v)] \Phi(v) dv \right\} \sigma_{s0} = \\ &= S_s \left[ 1 + c \int_0^t \exp(-i\omega v) \Phi(v) dv \right] \sigma_s^*(t) \end{aligned} \quad (21)$$

where  $i = \sqrt{-1}$  is an imaginary unit,  $( )^*$  denotes a complex quantity, and  $\omega \geq 0$  is a circular frequency. For  $t \rightarrow \infty$  we obtain a steady-state (harmonic) response in strains, i.e.

$$\begin{aligned} \varepsilon_s^*(t) &= S_s \left[ 1 + c \int_0^\infty \exp(-i\omega v) \Phi(v) dv \right] \sigma_s^*(t) \\ &= S_s^*(\omega) \sigma_s^*(t) \end{aligned} \quad (22)$$

where

$$\begin{aligned} S_s^*(\omega) &= S_s [1 + c \bar{\Phi}(i\omega)] = S_s \left[ 1 + c \frac{1}{1 + (i\omega T_c)^r} \right] = S_s'(\omega) + i S_s''(\omega) \\ S_s'(\omega) &= S_s \left[ 1 + c \frac{1 + (\omega T_c)^r \cos(\pi r/2)}{1 + 2(\omega T_c)^r \cos(\pi r/2) + (\omega T_c)^{2r}} \right] \\ S_s''(\omega) &= -S_s c \frac{(\omega T_c)^r \sin(\pi r/2)}{1 + 2(\omega T_c)^r \cos(\pi r/2) + (\omega T_c)^{2r}} \end{aligned} \quad (23)$$

with the following interpretation and nomenclature:  $S_s^*(\omega)$  – complex shear compliance,  $S_s'(\omega)$  – shear storage compliance (real part of the complex shear compliance),  $S_s''(\omega)$  – shear loss compliance (imaginary part of the complex shear compliance).

The shear storage and loss compliances have the following properties:

$$\begin{aligned} S_s'(\omega) &> 0, \quad S_s'(0) = (1+c) S_s, \quad \lim_{\omega \rightarrow \infty} S_s'(\omega) = S_s \\ S_s''(\omega) &< 0 \text{ for } \omega > 0, \quad S_s''(0) = 0, \quad \lim_{\omega \rightarrow \infty} S_s''(\omega) = 0 \end{aligned} \quad (24)$$

Unconjugated inverse constitutive equations of the linear elasticity and viscoelasticity of a thermoset can be formulated using the Laplace transformation on Eqn. (10)<sub>1</sub> (Eqn. (18)), i.e.

$$\begin{aligned}\bar{\sigma}_s(q) &= C_s \frac{1}{1+c\bar{\Phi}(q)} \bar{\epsilon}_s(q) \\ \frac{1}{1+c\bar{\Phi}(q)} &= \frac{1}{1+c\frac{1}{1+(qT_d)^r}} = 1 - d \frac{1}{1+(qT_d)^r} = 1 - d \bar{\Psi}(q) \quad (25) \\ d &= \frac{c}{1+c}, \quad T_d^r = \frac{1}{1+c} T_c^r, \quad \bar{\Psi}(q) = \frac{1}{1+(qT_d)^r}\end{aligned}$$

After inserting Eqn. (25)<sub>2</sub> into Eqn. (25)<sub>1</sub>, we get

$$\bar{\sigma}_s(q) = C_s [1 - d \bar{\Psi}(q)] \bar{\epsilon}_s(q) \quad (26)$$

The inverse Laplace transformation on Eqn. (26) gives the first unconjugated inverse constitutive equation. A full set of unconjugated inverse constitutive equations of the linear elasticity and viscoelasticity of a thermoset has the form:

$$\begin{aligned}\sigma_s(t) &= \tilde{C}_s(t) \otimes \epsilon_s(t), \quad \sigma_b(t) = C_b \epsilon_b(t) \\ \tau(t) &= \tilde{C}_s(t) \otimes \delta(t)\end{aligned} \quad (27)$$

where

$$\begin{aligned}\tilde{C}_s(t) &= C_s \left[ 1 - d \int_0^t \Psi(v) dv \right], \\ \Psi(t) &= \frac{1}{T_d} \int_0^\infty \exp\left(-\frac{ut}{T_d}\right) u L(u) du\end{aligned} \quad (28)$$

with the following interpretation and nomenclature:  $T_d$  – relaxation time,  $\Psi(t)$  – generic function for shear strains (Mittag-Leffler fractional exponential function),  $\tilde{C}_s(t)$  – elastic-viscoelastic shear stiffness,  $d$  – long-term relaxation coefficient.

Equations (11)<sub>2</sub> and (28)<sub>2</sub> are analogous; hence, generic function  $\Psi(t)$  has properties analogous to those of generic function  $\Phi(t)$  (Eqns. (16)<sub>4-6</sub>).

Viscoelastic parameters  $d, T_d$  are expressed in terms of previous viscoelastic parameters  $c, T_c$ , as specified in Eqns. (25)<sub>3,4</sub>. From Eqn. (25)<sub>4</sub> we obtain

$$T_d = T_c \exp\left(\frac{1}{r} \ln \frac{1}{1+c}\right) \quad (29)$$

The transformations of Eqn. (27)<sub>1</sub> are analogous as for Eq. (10)<sub>1</sub>. The results are collected below:

1) Equation (27)<sub>1</sub> in a fully explicit form:

$$\sigma_s(t) = C_s \left[ \epsilon_s(t) - d \int_0^t \Psi(v) \epsilon_s(t-v) dv \right] \quad (30)$$

2) relaxation function:

$$\psi(t) = \int_0^t \Psi(v) dv = 1 - \int_0^\infty \exp\left(-\frac{ut}{T_d}\right) L(u) du \quad (31)$$

3) stress response to the strain programme for relaxation:

$$\epsilon_s(t) = \epsilon_{s0} H(t) \Rightarrow \sigma_s(t) = C_s [1 - d \psi(t)] \epsilon_{s0} \quad (32)$$

4) Laplace transform of the generic function for strains:

$$\bar{\Psi}(q) = \int_0^\infty \exp(-qt) \Psi(t) dt = \frac{1}{1+(qT_d)^r} \quad (33)$$

5) harmonic strain programme:

$$\epsilon_s^*(t) = \epsilon_{s0} \exp(i\omega t) \quad (34)$$

6) steady-state (harmonic) elastic-viscoelastic response in stress to the harmonic strain programme:

$$\sigma_s^*(t) = C_s^*(\omega) \epsilon_s^*(t) \quad (35)$$

where

$$\begin{aligned}C_s^*(\omega) &= C_s [1 - d \bar{\Psi}(i\omega)] = C_s \left[ 1 - d \frac{1}{1 + (i\omega T_d)^r} \right] = C_s'(\omega) + i C_s''(\omega) \\ C_s'(\omega) &= C_s \left[ 1 - d \frac{1 + (\omega T_d)^r \cos(\pi r/2)}{1 + 2(\omega T_d)^r \cos(\pi r/2) + (\omega T_d)^{2r}} \right] \\ C_s''(\omega) &= C_s d \frac{(\omega T_d)^r \sin(\pi r/2)}{1 + 2(\omega T_d)^r \cos(\pi r/2) + (\omega T_d)^{2r}}\end{aligned} \quad (36)$$

with the following interpretation and nomenclature:  $C_s^*(\omega)$  – complex shear stiffness,  $C_s'(\omega)$  – shear storage stiffness (real part of the complex shear stiffness),  $C_s''(\omega)$  – shear loss stiffness (imaginary part of the complex shear stiffness).

The shear storage and loss stiffnesses have properties analogous to Eqns. (24), i.e.

$$\begin{aligned}C_s'(\omega) &> 0, \quad C_s'(0) = (1-d) C_s, \quad \lim_{\omega \rightarrow \infty} C_s'(\omega) = C_s \\ C_s''(\omega) &> 0 \text{ for } \omega > 0, \quad C_s''(0) = 0, \quad \lim_{\omega \rightarrow \infty} C_s''(\omega) = 0\end{aligned} \quad (37)$$

The results for the limit case  $r \rightarrow 1$  are summarized in Appendix A.

Conjugated constitutive equations of the linear elasticity and viscoelasticity of a thermoset result from Eqns. (2-12, 27, 28, 30) and have the form:

- standard equations:

$$\epsilon(t) = \tilde{S}(t) \otimes \sigma(t), \quad \delta(t) = \tilde{S}_s(t) \otimes \tau(t) \quad (38)$$

- inverse equations:

$$\sigma(t) = \tilde{C}(t) \otimes \epsilon(t), \quad \tau(t) = \tilde{C}_s(t) \otimes \delta(t) \quad (39)$$

where

$$\tilde{S}(t) = \tilde{S}_s(t) (\mathbf{I} - \mathbf{A}) + S_b \mathbf{A}, \quad \tilde{C}(t) = \tilde{C}_s(t) (\mathbf{I} - \mathbf{A}) + C_b \mathbf{A} \quad (40)$$

are the elastic-viscoelastic compliance and stiffness matrices of a homogeneous isotropic material. Equations (38, 39) can be rewritten in the following equivalent form:

- standard equations:

$$\begin{aligned}\epsilon(t) &= \mathbf{S} \sigma(t) + S_s c \int_0^t \Phi(v) dv (\mathbf{I} - \mathbf{A}) \otimes \sigma(t) \\ \delta(t) &= S_s \tau(t) + S_s c \int_0^t \Phi(v) dv \otimes \tau(t)\end{aligned} \quad (41)$$

- inverse equations:

$$\begin{aligned}\sigma(t) &= \mathbf{C} \epsilon(t) - C_s d \int_0^t \Psi(v) dv (\mathbf{I} - \mathbf{A}) \otimes \epsilon(t) \\ \tau(t) &= C_s \delta(t) - C_s d \int_0^t \Psi(v) dv \otimes \delta(t)\end{aligned} \quad (42)$$

## MODELLING OF LINEAR ELASTICITY AND VISCOELASTICITY OF UFRT COMPOSITES

Each lamina in a laminate shell reinforced with a stitched fabric is a UFRT composite. The assumptions made for modelling the elasticity and viscoelasticity of UFRT composites are given in the second section.

The  $x_1x_2x_3$  local Cartesian coordinate system has the following position:  $x_1$  – direction of reinforcing fibres,  $x_2$  – in-plane direction,  $x_3$  – out-of-plane direction. Plane  $x_2x_3$  is a transverse isotropy plane.

The stress and strain components in the  $x_1x_2x_3$  system are specified in Section 3. Consider the general case of an orthotropic solid body. The orthotropic linear elastic-brittle properties of this body are described by 12 effective elastic constants, 9 effective strength constants and 9 effective ultimate strain constants, i.e.  $E_1, E_2, E_3$  – Young's modules,  $\nu_{23}, \nu_{13}, \nu_{12}, \nu_{32}, \nu_{31}, \nu_{21}$  – Poisson's ratios,  $G_{23}, G_{13}, G_{12}$  – shear modules,  $R_{1t}, R_{2t}, R_{3t}$  – tensile strengths,  $R_{1c}, R_{2c}, R_{3c}$  – compressive strengths,  $R_{23}, R_{13}, R_{12}$  – shear strengths,  $e_{1t}, e_{2t}, e_{3t}$  – ultimate normal strains under tension,  $e_{1c}, e_{2c}, e_{3c}$  – ultimate normal strains under compression,  $g_{23}, g_{13}, g_{12}$  – ultimate shear strains. Poisson's ratios meet the following well-known conditions (Ref. [1]):

$$\nu_{jk}/E_k = \nu_{kj}/E_j, \quad j, k = 1, 2, 3, \quad j \neq k \quad (43)$$

Ratio  $\nu_{jk}$  determines contraction in the  $j$ -direction when stretching in the  $k$ -direction.

A monotropic solid body meets the following symmetry conditions:

$$\begin{aligned} E_3 &= E_2, \quad \nu_{13} = \nu_{12}, \quad G_{13} = G_{12} \\ G_{23} &= E_2/2(1 + \nu_{32}) \\ R_{3t} &= R_{2t}, \quad R_{3c} = R_{2c}, \quad R_{13} = R_{12} \\ e_{3t} &= e_{2t}, \quad e_{3c} = e_{2c}, \quad g_{13} = g_{12} \end{aligned} \quad (44)$$

In the linear elastic range, a UFRT composite modelled as a monotropic material is described by 5 independent elastic constants, i.e.  $E_1, E_2, \nu_{21}, \nu_{32}, G_{12}$ , where:  $E_1$  – longitudinal Young's modulus,  $E_2$  – transverse Young's modulus,  $\nu_{21}$  – greater Poisson's ratio in the longitudinal plane,  $\nu_{32}$  – Poisson's ratio in the transverse plane,  $G_{12}$  – shear modulus in the longitudinal plane. The elastic shear and bulk modules corresponding to the transverse isotropy plane are

$$G_{23} = E_2/2(1 + \nu_{32}), \quad B_2 = E_2/3(1 - 2\nu_{32}) \quad (45)$$

Conjugated standard constitutive equations of the linear elasticity of a homogenized UFRT composite are well-known (Ref. [1]) and can be written in the form

$$\boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\sigma}, \quad \boldsymbol{\delta} = \{\mathbf{S}\}\boldsymbol{\tau} \quad (46)$$

where

$$\begin{aligned} \mathbf{S} &= \begin{bmatrix} S_{11} & S_{12} & S_{12} \\ S_{12} & S_{22} & S_{23} \\ S_{12} & S_{23} & S_{22} \end{bmatrix}, \quad \{\mathbf{S}\} = \begin{bmatrix} S_{s4} & 0 & 0 \\ 0 & S_{s5} & 0 \\ 0 & 0 & S_{s5} \end{bmatrix} \\ S_{11} &= 1/E_1, \quad S_{22} = 1/E_2, \quad S_{12} = -\nu_{21}/E_1, \quad S_{23} = -\nu_{32}/E_2 \\ S_{s4} &= 1/2G_{23} = S_{22} - S_{23}, \quad S_{s5} = 1/2G_{12} \end{aligned} \quad (47)$$

Vectors  $\boldsymbol{\sigma}, \boldsymbol{\tau}, \boldsymbol{\varepsilon}, \boldsymbol{\delta}$  are defined in Eqns. (3)<sub>1-4</sub>. Matrices  $\mathbf{S}, \{\mathbf{S}\}$  are elastic compliance matrices corresponding to the normal/shear strains for a homogeneous monotropic material.

Conjugated inverse constitutive equations of the linear elasticity of a homogenized UFRT composite are also well-known (Ref. [1]) and can be written in the form

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}, \quad \boldsymbol{\tau} = \{\mathbf{C}\}\boldsymbol{\delta} \quad (48)$$

where

$$\begin{aligned} \mathbf{C} = \mathbf{S}^{-1} &= \begin{bmatrix} C_{11} & C_{12} & C_{12} \\ C_{12} & C_{22} & C_{23} \\ C_{12} & C_{23} & C_{22} \end{bmatrix}, \quad \{\mathbf{C}\} = \{\mathbf{S}\}^{-1} = \begin{bmatrix} C_{s4} & 0 & 0 \\ 0 & C_{s5} & 0 \\ 0 & 0 & C_{s5} \end{bmatrix} \\ C_{11} &= E_1(1 - \nu_{32}^2)/\Delta, \quad C_{22} = E_2(1 - \nu_{12}\nu_{21})/\Delta \\ C_{12} &= E_2\nu_{21}(1 + \nu_{32})/\Delta, \quad C_{23} = E_2(\nu_{32} + \nu_{12}\nu_{21})/\Delta \\ C_{s4} &= 2G_{23}, \quad C_{s5} = 2G_{12} \\ \Delta &= 1 - \nu_{32}^2 - 2\nu_{21}\nu_{12}(1 + \nu_{32}) \end{aligned} \quad (49)$$

Matrices  $\mathbf{C}, \{\mathbf{C}\}$  are elastic stiffness matrices corresponding to the normal/shear strains for a homogeneous monotropic material.

Based on Ref. [41], unconjugated standard constitutive equations of the linear elasticity of a homogenized UFRT composite, equivalent to Eqn. (46)<sub>1</sub>, are sought in the following form:

$$\boldsymbol{\varepsilon}_s = \{\mathbf{S}_s\}\boldsymbol{\sigma}_s, \quad \boldsymbol{\varepsilon}_b = \{\mathbf{S}_b\}\boldsymbol{\sigma}_b \quad (50)$$

where

$$\begin{aligned} \boldsymbol{\varepsilon}_s &= (\mathbf{I} - \mathbf{B})\boldsymbol{\varepsilon}, \quad \boldsymbol{\sigma}_s = (\mathbf{I} - \mathbf{A})\boldsymbol{\sigma}, \quad \boldsymbol{\varepsilon}_b = \mathbf{B}\boldsymbol{\varepsilon}, \quad \boldsymbol{\sigma}_b = \mathbf{A}\boldsymbol{\sigma} \\ \{\mathbf{S}_s\} &= \text{diag}(S_{s1}, S_{s2}, S_{s2}), \quad \{\mathbf{S}_b\} = \text{diag}(S_{b1}, S_{b2}, S_{b2}) \\ \mathbf{I} &= \text{diag}(1, 1, 1), \quad \mathbf{A} = \frac{1}{3} \begin{bmatrix} 1 & 1/\lambda & 1/\lambda \\ \lambda & 1 & 1 \\ \lambda & 1 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \end{aligned} \quad (51)$$

with

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_s + \boldsymbol{\varepsilon}_b, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}_s + \boldsymbol{\sigma}_b \quad (52)$$

The following nomenclature and interpretation are introduced:  $\boldsymbol{\varepsilon}_s, \boldsymbol{\varepsilon}_b, \boldsymbol{\sigma}_s, \boldsymbol{\sigma}_b$  – vectors of the elastic quasi-shear/quasi-bulk strains/stresses in a monotropic material,  $\{\mathbf{S}_s\}, \{\mathbf{S}_b\}$  – elastic quasi-shear/quasi-bulk compliance matrices describing a monotropic material,  $\mathbf{A}, \mathbf{B}$  – transformation matrices.

Coefficient  $\lambda$  and compliances  $S_{s1}, S_{s2}, S_{b1}, S_{b2}$  result from the equivalence condition (52)<sub>1</sub>, i.e.

$$\mathbf{S}\boldsymbol{\sigma} = \{\mathbf{S}_s\}\boldsymbol{\sigma}_s + \{\mathbf{S}_b\}\boldsymbol{\sigma}_b = \{\mathbf{S}_s\}(\mathbf{I} - \mathbf{A})\boldsymbol{\sigma} + \{\mathbf{S}_b\}\mathbf{A}\boldsymbol{\sigma} \quad (53)$$

hence

$$\mathbf{S} = \{\mathbf{S}_s\}(\mathbf{I} - \mathbf{A}) + \{\mathbf{S}_b\}\mathbf{A} \quad (54)$$

The explicit form of Eqn. (54) is

$$\begin{aligned} \begin{bmatrix} 1/E_1 & -\nu_{21}/E_1 & -\nu_{21}/E_1 \\ -\nu_{21}/E_1 & 1/E_2 & -\nu_{32}/E_2 \\ -\nu_{21}/E_1 & -\nu_{32}/E_2 & 1/E_2 \end{bmatrix} &= \\ \frac{1}{3} \begin{bmatrix} 2S_{s1} + S_{b1} & (S_{b1} - S_{s1})/\lambda & (S_{b1} - S_{s1})/\lambda \\ \lambda(S_{b2} - S_{s2}) & 2S_{s2} + S_{b2} & S_{b2} - S_{s2} \\ \lambda(S_{b2} - S_{s2}) & S_{b2} - S_{s2} & 2S_{s2} + S_{b2} \end{bmatrix} & \end{aligned} \quad (55)$$



The solution of Eqn. (55) has the form

$$\begin{aligned} S_{s1} &= (1 + \nu_{21}\lambda)/E_1, & S_{s2} &= 1/2G_{23} \\ S_{b1} &= (1 - 2\nu_{21}\lambda)/E_1, & S_{b2} &= 1/3B_2 \\ \lambda &= \nu_{12}/\nu_{32}, & \nu_{12} &= \nu_{21}E_2/E_1 \end{aligned} \quad (56)$$

Note that  $S_{s2} = S_{s4}$ .

Unconjugated inverse constitutive equations of the linear elasticity of a homogenized UFRT composite, corresponding to Eqns. (50), have the form

$$\boldsymbol{\sigma}_s = \{\mathbf{C}_s\}\boldsymbol{\varepsilon}_s, \quad \boldsymbol{\sigma}_b = \{\mathbf{C}_b\}\boldsymbol{\varepsilon}_b \quad (57)$$

where

$$\begin{aligned} \{\mathbf{C}_s\} &= \text{diag}(C_{s1}, C_{s2}, C_{s2}), & \{\mathbf{C}_b\} &= \text{diag}(C_{b1}, C_{b2}, C_{b2}) \\ C_{s1} &= E_1/(1 + \nu_{21}\lambda), & C_{s2} &= 2G_{23} \\ C_{b1} &= E_1/(1 - 2\nu_{21}\lambda), & C_{b2} &= 3B_2 \end{aligned} \quad (58)$$

Note that  $C_{s2} = C_{s4}$ . Matrices  $\{\mathbf{C}_s\}$ ,  $\{\mathbf{C}_b\}$  are the elastic quasi-shear/quasi-bulk stiffness matrices describing a monotropic material.

Matrix  $\mathbf{B}$  can be determined from the identity condition (52)<sub>2</sub>, i.e.

$$\mathbf{C}\boldsymbol{\varepsilon} = \{\mathbf{C}_s\}\boldsymbol{\varepsilon}_s + \{\mathbf{C}_b\}\boldsymbol{\varepsilon}_b = \{\mathbf{C}_s\}(\mathbf{I} - \mathbf{B})\boldsymbol{\varepsilon} + \{\mathbf{C}_b\}\mathbf{B}\boldsymbol{\varepsilon} \quad (59)$$

hence

$$\mathbf{C} = \{\mathbf{C}_s\}(\mathbf{I} - \mathbf{B}) + \{\mathbf{C}_b\}\mathbf{B} \quad (60)$$

The analytical solution of the matrix equation (60) has the form

$$\mathbf{B} = \{\mathbf{D}\}(\mathbf{C} - \{\mathbf{C}_s\}) = \begin{bmatrix} B_{11} & B_{12} & B_{12} \\ B_{21} & B_{22} & B_{23} \\ B_{21} & B_{23} & B_{22} \end{bmatrix} \quad (61)$$

where  $\mathbf{B}$  is a dimensionless matrix and

$$\begin{aligned} \{\mathbf{D}\} &= \text{diag}(D_1, D_2, D_2), & D_1 &= 1/(C_{b1} - C_{s1}), \\ & & D_2 &= 1/(C_{b2} - C_{s2}) \end{aligned} \quad (62)$$

Unconjugated standard constitutive equations of the linear elasticity and viscoelasticity of a UFRT composite, corresponding to Eqns. (50), (46)<sub>2</sub>, are predicted in the following form:

$$\begin{aligned} \boldsymbol{\varepsilon}_s(t) &= \{\tilde{\mathbf{S}}_s(t)\} \otimes \boldsymbol{\sigma}_s(t), & \boldsymbol{\varepsilon}_b(t) &= \{\mathbf{S}_b\}\boldsymbol{\sigma}_b(t) \\ \boldsymbol{\delta}(t) &= \{\tilde{\mathbf{S}}(t)\} \otimes \boldsymbol{\tau}(t) \end{aligned} \quad (63)$$

for time variable  $t \geq 0$ , with

$$\begin{aligned} \{\tilde{\mathbf{S}}_s(t)\} &= \text{diag}[\tilde{S}_{s1}(t), \tilde{S}_{s4}(t), \tilde{S}_{s4}(t)], \\ \{\tilde{\mathbf{S}}(t)\} &= \text{diag}[\tilde{S}_{s4}(t), \tilde{S}_{s5}(t), \tilde{S}_{s5}(t)] \\ \tilde{S}_{sj}(t) &= S_{sj} \left[ 1 + c_j \int_0^t \Phi(v) dv \right], & j &= 1, 4, 5 \\ \Phi(t) &= \frac{1}{T_c} \int_0^\infty \exp\left(-\frac{ut}{T_c}\right) u L(u) du \\ L(u) &= \frac{\sin \pi r}{\pi u} \frac{u^r}{1 + 2u^r \cos \pi r + u^{2r}}, & 0 < r < 1 \end{aligned} \quad (64)$$

The nomenclature and interpretation are as follows:  $L(u)$  – function defining the Mittag-Leffler fractional exponential function in an integral form,  $r$  – fraction

defining the Mittag-Leffler fractional exponential function,  $T_c$  – retardation time,  $\Phi(t)$  – generic function for shear stresses (Mittag-Leffler fractional exponential function),  $\{\tilde{\mathbf{S}}_s(t)\}$  – elastic-viscoelastic quasi-shear compliance matrix,  $\{\tilde{\mathbf{S}}(t)\}$  – elastic-viscoelastic shear compliance matrix,  $c_1, c_4, c_5$  – long-term creep coefficients,  $\otimes$  – convolution product operator.

The rheological model of a UFRT composite, governed by Eqns. (63, 64), is described by 5 elastic constants  $E_1, E_2, \nu_{21}, \nu_{32}, G_{12}$  and 5 viscoelastic constants  $r, T_c, c_1, c_4, c_5$ . Constants  $r, T_c$  are determined experimentally for a thermoset and common for the polymer matrix and the UFRT composite. An algorithm for determining constants  $c_1, c_4, c_5$ , based on the viscoelastic-elastic correspondence principle, was formulated in Part 2 [40].

Further transformations on Eqns. (63)<sub>1,3</sub> (6 equations in an explicit form) will be presented on the first equation contained in matrix equation (63)<sub>1</sub>, i.e.

$$\varepsilon_{s1}(t) = \tilde{S}_{s1}(t) \otimes \sigma_{s1}(t), \quad \tilde{S}_{s1}(t) = S_{s1} \left[ 1 + c_1 \int_0^t \Phi(v) dv \right] \quad (65)$$

The transformations are analogous to those for a thermoset as presented in Section 3. The results appear as:

1) equation (65)<sub>1</sub> in the fully explicit form:

$$\varepsilon_{s1}(t) = S_{s1} \left[ \sigma_{s1}(t) + c_1 \int_0^t \Phi(v) \sigma_{s1}(t-v) dv \right] \quad (66)$$

2) stress programme for quasi-shear creep:

$$\sigma_{s1}(t) = \sigma_{s1,0} H(t) \quad (67)$$

where  $\sigma_{s1,0}$  is the initial quasi-shear stress;

3) quasi-shear strain response to the quasi-shear stress programme (67):

$$\varepsilon_{s1}(t) = S_{s1} [1 + c_1 \varphi(t)] \sigma_{s1,0} \quad (68)$$

4) creep function  $\varphi(t)$  defined by Eqn. (15);

5) Laplace transform made on Eqn. (65)<sub>1</sub>:

$$\bar{\varepsilon}_{s1}(q) = S_{s1} [1 + c_1 \bar{\Phi}(q)] \bar{\sigma}_{s1}(q) \quad (69)$$

where Laplace transform  $\bar{\Phi}(q)$  is defined by Eqn. (19)<sub>3</sub> and

$$\begin{aligned} \bar{\varepsilon}_{s1}(q) &= \int_0^\infty \exp(-qt) \varepsilon_{s1}(t) dt, \\ \bar{\sigma}_{s1}(q) &= \int_0^\infty \exp(-qt) \sigma_{s1}(t) dt \end{aligned} \quad (70)$$

6) harmonic strain programme:

$$\sigma_{s1}^*(t) = \sigma_{s1,0} \exp(i\omega t) \quad (71)$$

7) steady-state (harmonic) elastic-viscoelastic response in strains to the harmonic stress programme:

$$\begin{aligned} \varepsilon_{s1}^*(t) &= S_{s1} \left[ 1 + c_1 \int_0^\infty \exp(-i\omega v) \Phi(v) dv \right] \sigma_{s1}^*(t) = \\ &S_{s1}^*(\omega) \sigma_{s1}^*(t) \end{aligned} \quad (72)$$

where

$$S_{s1}^*(\omega) = S_{s1} [1 + c_1 \bar{\Phi}(i\omega)]$$

$$\begin{aligned}
&= S_{s1} \left[ 1 + c_1 \frac{1}{1 + (i\omega T_c)^r} \right] = S'_{s1}(\omega) + iS''_{s1}(\omega) \\
S'_{s1}(\omega) &= S_{s1} \left[ 1 + c_1 \frac{1 + (\omega T_c)^r \cos(\pi r/2)}{1 + 2(\omega T_c)^r \cos(\pi r/2) + (\omega T_c)^{2r}} \right] \\
S''_{s1}(\omega) &= -S_{s1} c_1 \frac{(\omega T_c)^r \sin(\pi r/2)}{1 + 2(\omega T_c)^r \cos(\pi r/2) + (\omega T_c)^{2r}}
\end{aligned} \quad (73)$$

with the following interpretation and nomenclature:  $S_{s1}^*(\omega)$  – first complex quasi-shear compliance,  $S'_{s1}(\omega)$  – first quasi-shear storage compliance,  $S''_{s1}(\omega)$  – first quasi-shear loss compliance.

The quasi-shear storage and loss compliances have properties analogous to those described by Eqns. (24).

By applying transformations analogous to those for Eqns. (25, 26, 29), we get unconjugated inverse constitutive equations of the linear elasticity and viscoelasticity of a UFRT composite in the following form:

$$\begin{aligned}
\boldsymbol{\sigma}_s(t) &= \{\tilde{\mathbf{C}}_s(t)\} \otimes \boldsymbol{\varepsilon}_s(t), \quad \boldsymbol{\sigma}_b(t) = \{\mathbf{C}_b\} \boldsymbol{\varepsilon}_b(t) \\
\boldsymbol{\tau}(t) &= \{\tilde{\mathbf{C}}(t)\} \otimes \boldsymbol{\delta}(t)
\end{aligned} \quad (74)$$

for time variable  $t \geq 0$ , with

$$\begin{aligned}
\{\tilde{\mathbf{C}}_s(t)\} &= \text{diag}[\tilde{C}_{s1}(t), \tilde{C}_{s4}(t), \tilde{C}_{s5}(t)] \\
\{\tilde{\mathbf{C}}(t)\} &= \text{diag}[\tilde{C}_{s4}(t), \tilde{C}_{s5}(t), \tilde{C}_{s5}(t)] \\
\tilde{C}_{sj}(t) &= C_{sj} \left[ 1 - d_j \int_0^t \Psi_j(v) dv \right], \quad j = 1, 4, 5 \\
\Psi_j(t) &= \frac{1}{T_{dj}} \int_0^\infty \exp\left(-\frac{ut}{T_{dj}}\right) u L(u) du, \quad j = 1, 4, 5 \\
d_j &= \frac{c_j}{1 + c_j}, \quad T_{dj} = T_c \exp\left(\frac{1}{r} \ln \frac{1}{1 + c_j}\right), \quad j = 1, 4, 5
\end{aligned} \quad (75)$$

and the following nomenclature and interpretation:  $T_{dj}$  –  $j$ -th relaxation time,  $\Psi_j(t)$  –  $j$ -th shear strain generic function,  $\{\tilde{\mathbf{C}}_s(t)\}$  – elastic-viscoelastic quasi-shear stiffness matrix,  $\{\tilde{\mathbf{C}}(t)\}$  – elastic-viscoelastic shear stiffness matrix,  $d_1, d_4, d_5$  – long-term relaxation coefficients. Viscoelastic parameters  $d_j, T_{dj}$ ,  $j = 1, 4, 5$ , are expressed in terms of previous viscoelastic parameters  $c_j, T_c$ ,  $j = 1, 4, 5$ , as specified in Eqns. (75)<sub>5,6</sub>.

The transformations on Eqns. (74)<sub>1,3</sub> (6 equations in the explicit form) will be presented on the first equation contained in matrix equation (74)<sub>1</sub>, i.e.

$$\sigma_{s1}(t) = \tilde{C}_{s1}(t) \otimes \varepsilon_{s1}(t) \quad (76)$$

The results are collected below:

1) Equation (76) in the fully explicit form:

$$\sigma_{s1}(t) = C_{s1} \left[ \varepsilon_{s1}(t) - d_1 \int_0^t \Psi_1(v) \varepsilon_{s1}(t-v) dv \right] \quad (77)$$

2) relaxation functions:

$$\Psi_j(t) = \int_0^t \Psi_j(v) dv = 1 - \int_0^\infty \exp\left(-\frac{ut}{T_{dj}}\right) L(u) du, \quad j = 1, 2, 4, 5 \quad (78)$$

3) stress response to the strain programme for relaxation:

$$\varepsilon_{s1}(t) = \varepsilon_{s1,0} H(t) \Rightarrow \sigma_{s1}(t) = C_{s1} [1 - d_1 \psi_1(t)] \varepsilon_{s1,0} \quad (79)$$

4) Laplace transform of the generic function for strains:

$$\bar{\Psi}_1(q) = \int_0^\infty \exp(-qt) \Psi_1(t) dt = \frac{1}{1 + (qT_{d1})^r} \quad (80)$$

5) harmonic strain programme:

$$\varepsilon_{s1}^*(t) = \varepsilon_{s1,0} \exp(i\omega t) \quad (81)$$

6) steady-state (harmonic) elastic-viscoelastic response in stress to the harmonic strain programme:

$$\sigma_{s1}^*(t) = C_{s1}^*(\omega) \varepsilon_{s1}^*(t) \quad (82)$$

where

$$\begin{aligned}
C_{s1}^*(\omega) &= C_{s1} [1 - d_1 \bar{\Psi}_1(i\omega)] \\
&= C_{s1} \left[ 1 - d_1 \frac{1}{1 + (i\omega T_{d1})^r} \right] = C'_{s1}(\omega) + iC''_{s1}(\omega) \\
C'_{s1}(\omega) &= C_{s1} \left[ 1 - d_1 \frac{1 + (\omega T_{d1})^r \cos(\pi r/2)}{1 + 2(\omega T_{d1})^r \cos(\pi r/2) + (\omega T_{d1})^{2r}} \right] \\
C''_{s1}(\omega) &= C_{s1} d_1 \frac{(\omega T_{d1})^r \sin(\pi r/2)}{1 + 2(\omega T_{d1})^r \cos(\pi r/2) + (\omega T_{d1})^{2r}}
\end{aligned} \quad (83)$$

with the following interpretation and nomenclature:  $C_{s1}^*(\omega)$  – first complex quasi-shear stiffness,  $C'_{s1}(\omega)$  – first quasi-shear storage stiffness,  $C''_{s1}(\omega)$  – first quasi-shear loss stiffness. The quasi-shear storage and loss stiffness quantities have properties analogous to Eqns. (37).

Conjugated constitutive equations of the linear elasticity and viscoelasticity of a UFRT composite result from Eqns. (46-64, 74, 75) and have the form:

- standard equations:

$$\boldsymbol{\varepsilon}(t) = \tilde{\mathbf{S}}(t) \otimes \boldsymbol{\sigma}(t), \quad \boldsymbol{\delta}(t) = \{\tilde{\mathbf{S}}(t)\} \otimes \boldsymbol{\tau}(t) \quad (84)$$

- inverse equations:

$$\boldsymbol{\sigma}(t) = \tilde{\mathbf{C}}(t) \otimes \boldsymbol{\varepsilon}(t), \quad \boldsymbol{\tau}(t) = \{\tilde{\mathbf{C}}(t)\} \otimes \boldsymbol{\delta}(t) \quad (85)$$

where

$$\tilde{\mathbf{S}}(t) = \{\tilde{\mathbf{S}}_s(t)\} (\mathbf{I} - \mathbf{A}) + \{\mathbf{S}_b\} \mathbf{A}, \quad \tilde{\mathbf{C}}(t) = \{\tilde{\mathbf{C}}_s(t)\} (\mathbf{I} - \mathbf{B}) + \{\mathbf{C}_b\} \mathbf{B} \quad (86)$$

are the elastic-viscoelastic compliance and stiffness matrices of a homogeneous monotropic material. Transformation matrices  $\mathbf{A}, \mathbf{B}, \mathbf{I}$  are defined by Eqns. (51)<sub>7-9</sub>, (56)<sub>5</sub>, (61, 62).

Equations (84, 85) can be rewritten in the following equivalent form:

- standard equations:

$$\begin{aligned}
\boldsymbol{\varepsilon}(t) &= \mathbf{S} \boldsymbol{\sigma}(t) + \{\tilde{\mathbf{S}}_{sv}(t)\} (\mathbf{I} - \mathbf{A}) \otimes \boldsymbol{\sigma}(t), \\
\boldsymbol{\delta}(t) &= \{\mathbf{S}\} \boldsymbol{\tau}(t) + \{\tilde{\mathbf{S}}_v(t)\} \otimes \boldsymbol{\tau}(t) \\
\{\tilde{\mathbf{S}}_{sv}(t)\} &= \text{diag}[\tilde{S}_{sv1}(t), \tilde{S}_{sv4}(t), \tilde{S}_{sv4}(t)] \\
\{\tilde{\mathbf{S}}_v(t)\} &= \text{diag}[\tilde{S}_{sv4}(t), \tilde{S}_{sv5}(t), \tilde{S}_{sv5}(t)]
\end{aligned} \quad (87)$$

$$\tilde{S}_{svj}(t) = S_{sj}c_j \int_0^t \Phi(v) dv, \quad j = 1, 4, 5$$

- inverse equations:

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathbf{C} \boldsymbol{\varepsilon}(t) + \{\tilde{\mathbf{C}}_{sv}(t)\}(\mathbf{I} - \mathbf{B}) \otimes \boldsymbol{\varepsilon}(t) \\ \boldsymbol{\tau}(t) &= \{\mathbf{C}\} \boldsymbol{\delta}(t) + \{\tilde{\mathbf{C}}_v(t)\} \otimes \boldsymbol{\delta}(t) \\ \{\tilde{\mathbf{C}}_{sv}(t)\} &= \text{diag}[\tilde{C}_{sv1}(t), \tilde{C}_{sv4}(t), \tilde{C}_{sv4}(t)] \\ \{\tilde{\mathbf{C}}_v(t)\} &= \text{diag}[\tilde{C}_{sv4}(t), \tilde{C}_{sv5}(t), \tilde{C}_{sv5}(t)] \\ \tilde{C}_{svj}(t) &= -C_{sj}d_j \int_0^t \Psi_j(v) dv, \quad j = 1, 4, 5 \end{aligned} \quad (88)$$

Functions  $\Phi(v)$ ,  $\Psi_j(v)$ ,  $j = 1, 4, 5$ , are defined by Eqns. (64)<sub>4,5</sub>, (75)<sub>4,5</sub>. In Eqns. (87, 88), the separation of the elastic part from the viscoelastic part is visible.

For control purposes, consider the case of an isotropic material for which

$$\begin{aligned} E_j &= E, \quad \nu_{jk} = \nu, \quad G_{jk} = G = E/2(1 + \nu) \\ B_j &= B = E/3(1 - 2\nu), \quad j, k = 1, 2, 3 \end{aligned} \quad (89)$$

After inserting Eqns. (63) into Eqns. (49, 55, 56, 58, 62), we get

$$\begin{aligned} \lambda &= 1 \\ S_{s1} &= S_{s2} = 1/2G, \quad S_{b1} = S_{b2} = 1/3B \\ C_{s1} &= C_{s2} = 2G, \quad C_{b1} = C_{b2} = 3B \\ D_1 &= D_2 = (1 + \nu)(1 - 2\nu)/3E\nu \\ \mathbf{A} &= \mathbf{B} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{aligned} \quad (90)$$

The results (90) are consistent with the theory of linear elasticity of an isotropic solid body (Eqns. (9)).

## CONCLUSIONS

Advanced analytical modelling of the linear elasticity and viscoelasticity of thermosets and unidirectional fibre-reinforced thermoset-matrix (UFRT) composites was developed. In practice, a UFRT composite is a single lamina within a laminate made of a thermoset matrix reinforced with stitched glass fabrics. The components of a UFRT composite were assumed to be homogeneous linear viscoelastic (matrix)/elastic (fibre) isotropic materials. The homogenized UFRT composite was a linear viscoelastic monotropic material.

New rheological models (coded H-R/H) for thermosets and UFRT composites, described by the smallest possible number of the material constants, were developed. The generic function for viscoelastic shear/quasi-shear stresses in thermosets and UFRT composites was assumed as the Mittag-Leffler fractional exponential function in an integral form. The H-R/H model of a thermoset was described by two elastic and three viscoelastic parameters. The H-R/H model of the homogenized UFRT composite was described by five elastic and five viscoelastic parameters. Two viscoelastic parameters, i.e. retardation time, a fraction defining the viscoelasticity generic function, were common to the matrix and the composite.

The following constitutive equations of a thermoset were formulated:

- conjugated/unconjugated standard/inverse constitutive equations of linear elasticity;
- conjugated/unconjugated standard/inverse constitutive equations of linear elasticity-viscoelasticity.

The following constitutive equations of a homogenized UFRT composite were formulated:

- matrix conjugated/unconjugated standard/inverse constitutive equations of linear elasticity;
- matrix conjugated/unconjugated standard/inverse constitutive equations of linear elasticity-viscoelasticity.

The key tools for formulating the above equations are:

- complex shear compliance of a thermoset;
- complex shear stiffness of a thermoset;
- complex quasi-shear compliances of a homogenized UFRT composite;
- complex quasi-shear stiffnesses a homogenized UFRT composite.

The constitutive equations described quasi-static isothermal reversible elastic/viscoelastic processes. The solutions as well as the transformations transforming one quantity/equation into another are fully analytical. The conjugated constitutive equations of the linear elasticity-viscoelasticity of thermosets and UFRT composites can be implemented into commercial CAE systems such as LS-Dyna, MSC.Marc, and Abaqus.

## Acknowledgements

*Proofreading of the manuscript was done by Mrs. Christine Frank-Szarecka, Canada. This support is gratefully acknowledged.*

## APPENDIX A. CLASSIC RHEOLOGICAL FUNCTIONS

When  $r \rightarrow 1$  the generic functions become normal exponential functions (Ref. [42]), i.e.

$$L(u) = \delta(u - 1), \quad \Phi(t) = \frac{1}{T_c} \exp\left(-\frac{t}{T_c}\right), \quad \Psi(t) = \frac{1}{T_d} \exp\left(-\frac{t}{T_d}\right) \quad (A.1)$$

where  $\delta(u - 1)$  is a Dirac function. The final equations, useful for comparative purposes, are collected below:

- 1) creep/relaxation functions:

$$\varphi(t) = 1 - \exp(-t/T_c), \quad \psi(t) = 1 - \exp(-t/T_d) \quad (A.2)$$

- 2) time conversion:

$$T_d = T_c/(1 + c) \quad (A.3)$$

- 3) Laplace transforms of the generic functions:

$$\bar{\Phi}(q) = \frac{1}{1+q/T_c}, \quad \bar{\Psi}(q) = \frac{1}{1+q/T_d} \quad (A.4)$$

- 4) shear storage/loss compliances:

$$S'_s(\omega) = S_s \left[ 1 + c \frac{1}{1+(\omega T_c)^2} \right], \quad S''_s(\omega) = -S_s c \frac{\omega T_c}{1+(\omega T_c)^2} \quad (A.5)$$

5) shear storage/loss stiffnesses:

$$C'_s(\omega) = C_s \left[ 1 - d \frac{1}{1+(\omega T_d)^2} \right], \quad C''_s(\omega) = C_s d \frac{\omega T_d}{1+(\omega T_d)^2} \quad (\text{A.6})$$

Equations (A.1–A.5) describe a H-K/H rheological model of an isotropic material.

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