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COMPOSITES THEORY AND PRACTICE

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DISPERSION RELATIONS FOR COMPOSITE STRUCTURES. PART II. METHODS OF DETERMINING DISPERSION CURVES

In the first part of the current review, the fundamental assumptions of the theoretical model of elastic waves propagation in multilayered composite material are presented. Next, the equations which describe elastic wave motion in the case of single orthotropic lamina are derived. In the second part of this work, the most commonly used method of determining dispersion curves for multilayered composite material are discussed, namely: the transfer matrix method (TMM), global matrix method (GMM), stiffness matrix method (SMM) and finally the semi-analytical finite element method (SAFE). The first three methods are based on the relationships which are derived in the first part of this review. Moreover, TMM and GMM should be considered numerically unstable in the case of a relatively large product value of wave frequency and the total thickness of the composite plate. However, SMM seems to be unconditionally stable. The last method is based on the finite element approach and it can be used in order to confirm the results obtained using the analytical method. Finally, exemplary dispersion curves are presented. The dispersion curves are determined for the 8-th layer of the composite material, which is made of carbon fiber and epoxy resin. It is assumed that the wave front travels in an arbitrary direction.

Keywords: Lamb waves, composite materials, anisotropic layer, dispersion curves, phase velocity, group velocity

RÓWNANIA DYSPERSJI DLA STRUKTUR KOMPOZYTOWYCH. CZĘŚĆ II. METODY WYZNACZANIA KRZYWYCH DYSPERSJI

W części pierwszej pracy omówiono założenia dotyczące teoretycznego modelu propagacji fal sprężystych w wielowarstwowych materiałach kompozytowych. Następnie wyprowadzono równania opisujące zjawisko propagacji fal sprężystych w pojedynczej warstwie o ortotropowych własnościach mechanicznych. W części drugiej przedstawiono podstawy najczęściej wykorzystywanych metod wyznaczania krzywych dyspersji dla ośrodków wielowarstwowych, a mianowicie: *transfer matrix method (TMM), global matrix method (GMM), stiffness matrix method* (*SMM*), a także *semi-analytical finite element method (SAFE).* Pierwsze trzy podejścia oparte są bezpośrednio na równaniach wyprowadzonych w części pierwszej. Metody TMM oraz GMM uważane są za numerycznie niestabilne w przypadku odpowiednio dużych wartości iloczynu częstotliwości i całkowitej grubości płyty kompozytowej. Natomiast wydaje się, że podejście SMM jest numerycznie bezwarunkowo stabilne. Ostatnia z wymienionych metod oparta jest na metodzie elementów skończonych i można ją efektywnie wykorzystać w celu potwierdzenia wyników otrzymanych przy użyciu poprzednio wymienionych algorytmów. Jako przykład pokazano krzywe dyspersji wyznaczone dla 8-warstwowego materiału kompozytowego wykonanego z włókna węglowego, przy czym założono, że czoło fali porusza się w dowolnie założonym kierunku.

Słowa kluczowe: fale Lamba, materiały kompozytowe, warstwa anizotropowa, prędkość fazowa, prędkość grupowa

INTRODUCTION

In the case of multilayered composite materials, determining dispersion curves should be considered a challenging task. The difficulties are mainly caused by numerical instabilities, significant differences between the stiffness of the adjacent layers and the strong orthotropic mechanical properties of the layers. Chronologically, the first method is the transfer matrix method, also considered the simplest, however, in composite material, which consists of strongly orthotropic plies, the problem of numerical instabilities is especially visible. Better results can be obtained by using the global matrix method, on the other hand, this approach is not very efficient and it seems very complicated. In the case of materials with a large number of layers, obtaining a solution could be very time consuming. The latest and a relatively modern method, the stiffness matrix method, is considered numerically unconditionally stable. Nonetheless, according to the authors' experience, in the case of significant differences between the stiffness of the adjacent layers, determining the dispersion curves could be problematic. Finally, the semi-analytical method is also described. The main advantage of this approach is that it can be applied with the use of commercial software, which is based on the finite element method. Therefore, it can be used in order to confirm results obtained with the use of the above mentioned analytical method. To facilitate reading of further sections of this paper, the final relationships, namely equation (30), from the first part of the current review is rewritten below:

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \sigma_{33} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ V_1 & V_1 & V_3 & V_3 & V_5 & V_5 \\ W_1 & -W_1 & W_3 & -W_3 & W_5 & -W_5 \\ D_{11} & D_{11} & D_{13} & D_{13} & D_{15} & D_{15} \\ D_{21} & -D_{21} & D_{23} & -D_{23} & D_{25} & -D_{25} \\ D_{31} & -D_{31} & D_{33} & -D_{33} & D_{35} & -D_{35} \end{bmatrix} diag\left[e^{i\xi\alpha_j x_3}\right] \begin{Bmatrix} U_{11}e^{i\xi(x_1-ct)} \\ U_{12}e^{i\xi(x_1-ct)} \\ U_{13}e^{i\xi(x_1-ct)} \\ U_{14}e^{i\xi(x_1-ct)} \\ U_{15}e^{i\xi(x_1-ct)} \\ U_{16}e^{i\xi(x_1-ct)} \end{Bmatrix} . \tag{1}$$

This relationship should be considered the starting point in the discussion about the methods of determining dispersion curves.

TRANSFER MATRIX METHOD

According to the review by Lowe [1], the first paper devoted to deriving the equation of wave propagation in multilayered media was published by Thompson [2] in 1950. He introduced the so-called transfer matrix, which relates the displacement and stress at the top and bottom of the layer. The matrices for any number of isotropic layers could be coupled into one. Then, the dispersion curves can be obtained by applying appropriate stress boundary conditions. Thompson's approach was corrected in 1953 by Haskell [3]. However, this approach is limited to materials where all the layers are made of isotropic materials. Nayfeh [4, 5] extended the transfer matrix method to the case of composite materials, where the layers are made of anisotropic materials. Generally, relationship (1) can be written in the following form:

$$\{P_k\} = [X_k][D_k]\{U_k\}. \tag{2}$$

Note that elements of diagonal matrix $[D_k]$ depend on coordinate x'_3. In the case of the top surface of the k-th layer, where local coordinate $x'_3 = 0$, matrix $[D_k]$ is the identity matrix. Therefore, the above relation can be simplified:

$$\{P_k\}^{top} = [X_k]\{U_k\}. \tag{3}$$

The superscript 'top' denotes the top surface. Next, in the case of the bottom surface, superscript 'bot', where $x_3 = d_k$, relation (2) can be written as follows:

$$\{P_k\}^{bot} = [X_k][D_k]^{bot}\{U_k\}. \tag{4}$$

By combining relations (3) and (4), the relationship, which relates the displacement and stress on the top surface at $x'_3 = 0$ to those on the top surface at $x'_3 = d_k$, is obtained:

$$\{P_k\}^{bot} = [X_k][D_k]^{bot}[X_k]^{-1}\{P_k\}^{top} = [T_k]\{P_k\}^{top} \tag{5}$$

Matrix $[T_k]$ is called the transfer matrix for layer k. By applying the above procedure for each layer, it is possible to relate the displacements and the stresses at the top and bottom surface of the analyzed composite material. It can be done by multiplying particular transfer matrices:

$$[T] = \prod_{k=1}^{n}[T_k]. \tag{6}$$

According to Giurgiutiu [6], the total transfer matrix expression is:

$$\begin{Bmatrix} \{u^{bot}\} \\ \{\sigma^{bot}\} \end{Bmatrix} = \begin{bmatrix} [T_{uu}] & [T_{u\sigma}] \\ [T_{u\sigma}] & [T_{\sigma\sigma}] \end{bmatrix} \begin{Bmatrix} \{u^{top}\} \\ \{\sigma^{top}\} \end{Bmatrix} \tag{7}$$

To determine the dispersion curves, stress free boundaryconditions have to be applied. It leads to the characteristic equation:

$$\det([T_{u\sigma}]) = 0. \tag{8}$$

Unfortunately, TMM for higher frequencies and thicker plates reveals numerical instabilities (*fd* problem [6]), especially, in the case of strongly orthotropic materials. Thus this method, due to its relative simplicity, can be used only for estimating the frequency when a L_1, SV_1 or SH_1 wave mode appears. However, there are some works devoted to the improvement of numerical stability in the case of the Thomson-Haskell formulation. The papers by Castings and Hosten can be cited here [7-9]. The first computer programs for multilayered materials were created in the 1960s [10-12].

GLOBAL MATRIX METHOD

The global matrix formulation was proposed in 1964 by Knopoff [13]. At the beginning, this method was used in the case of isotropic layers (Lowe [1], Pavlakovic [14], Demcenko and Mazeika [15], Schwab [16], Schmidt and Tango [17]). However, now this method is also used in the case of composite materials (Pant et al. [18, 19]). It is worth mentioning here that commercial software DISPERSE [20] based on this method is available. In order to derive the GM formulation [18] we recall relation (2). For interface 2 (Fig. 5 in the first part of the review), which consists of the bottom surface of layer 1 and the top surface of layer 2, the displacement and stresses at the interface can be expressed as:

$$\{P_1\}^{bot} = [X_1]^{bot}[D_1]^{bot}\{U_1\}^{bot}, \quad \{P_2\}^{top} = [X_2]^{top}[D_2]^{top}\{U_2\}^{top} \tag{9}$$

Next, assuming that the displacements and stresses are continuous, the above relations can be rewritten in the following way [18]:

$$\left[[Z_1]^{bot} \; -[Z_2]^{top}\right]\begin{Bmatrix} \{U_1\}^{bot} \\ \{U_2\}^{top} \end{Bmatrix} = \{0\} \tag{10}$$

where $[Z_1]^{bot} = [X_1]^{bot}\ [D_1]^{bot}$, $[Z_2]^{top} = [X_2]^{top}\ [D_2]^{top}$. The above procedure can be repeated for all the layers to form the Global Matrix. Generally 6(n-1) equations with 6n unknowns for six partial waves are obtained. If the considered composite material is surrounded by vacuum and, for example, consists of 3 layers, the final form of the Global Matrix takes the form:

$$\begin{Bmatrix} \{P_{i1}\} \\ \{0\} \\ \{0\} \\ \{P_{i4}\} \end{Bmatrix} = \begin{bmatrix} [-Z_1]^{top} & [0] & [0] \\ [\ Z_1]^{bot} & [-Z_2]^{top} & [0] \\ [0] & [\ Z_2]^{bot} & [-Z_3]^{top} \\ [0] & [0] & [\ Z_3]^{bot} \end{bmatrix} \begin{Bmatrix} \{U_1\} \\ \{U_2\} \\ \{U_3\} \end{Bmatrix} \tag{11}$$

where $\{P_{i1}\}$, $\{P_{i2}\}$ are the vectors containing the displacement and stresses on the top and bottom surfaces of the composite material and subscript i means *interface*. For Lamb waves, the stress components are zero on these surfaces. Neglecting the displacement components from the top and bottom surfaces of (11), a new matrix is obtained, namely [19]:

$$\begin{Bmatrix} \{0\}^* \\ \{0\} \\ \{0\} \\ \{0\}^* \end{Bmatrix} = \begin{bmatrix} [-Z_1]^{top^*} & [0] & [0] \\ [\ Z_1]^{bot} & [-Z_2]^{top} & [0] \\ [0] & [\ Z_2]^{bot} & [-Z_3]^{top} \\ [0] & [0] & [\ Z_3]^{bot^*} \end{bmatrix} \begin{Bmatrix} \{U_1\} \\ \{U_2\} \\ \{U_3\} \end{Bmatrix}, \tag{12}$$

where:

$$[Z_1]^{top^*} = \begin{bmatrix} Z_{41} & Z_{42} & Z_{43} & Z_{44} & Z_{45} & Z_{46} \\ Z_{51} & Z_{52} & Z_{53} & Z_{54} & Z_{55} & Z_{56} \\ Z_{61} & Z_{62} & Z_{63} & Z_{64} & Z_{65} & Z_{66} \end{bmatrix}_1^{top},$$
$$[Z_3]^{bot^*} = \begin{bmatrix} Z_{41} & Z_{42} & Z_{43} & Z_{44} & Z_{45} & Z_{46} \\ Z_{51} & Z_{52} & Z_{53} & Z_{54} & Z_{55} & Z_{56} \\ Z_{61} & Z_{62} & Z_{63} & Z_{64} & Z_{65} & Z_{66} \end{bmatrix}_3^{bot}. \tag{13}$$

A nontrivial solution of (12) exists if the determinant of global coefficient matrix $[Z]$ is equal to zero:

$$\det([Z]) = 0. \tag{14}$$

It is worth pointing out here that the numerical solution of equation (14) needs an appropriate algorithm [21]. During the solution process matrix $[Z]$ is frequently close to being singular. Moreover, searching algorithms have to rely on precise estimation, using complex numbers, of the solution close to singularity in order to converge. In the case of determining dispersion curves, the whole process is repeated many times without failure. According Love [21], the most effective algorithm is the reduction part of the Gaussian elimination scheme.

STIFFNESS MATRIX METHOD

In order to avoid any numerical instabilities, which are the main disadvantages of TMM, Kausel [22], Wang and Rokhlin [23-25] introduced the Stiffness Matrix Method. In contrast to TMM, the Stiffness Matrix relates the stress components on the top and bottom surface of a particular layer to the displacement components on the top and bottom surface. This change makes this method unconditionally numerically stable and slightly more computationally efficient than the TMM method. Generally, SM can be written as [26]:

$$\begin{Bmatrix} \{\sigma\}_k^{top} \\ \{\sigma\}_k^{bot} \end{Bmatrix} = [K]_k = [A]_k [B]_k^{-1} =$$
$$\begin{bmatrix} [D]^{top} & [D]^{bot}[H]^{bot} \\ [D]^{top}[H]^{top} & [D]^{bot} \end{bmatrix} \begin{bmatrix} [P_S]^{top} & [P_S]^{bot}[H]^{bot} \\ [P_S]^{top}[H]^{top} & [P_S]^{bot} \end{bmatrix}^{-1} \begin{Bmatrix} \{u\}_k^{top} \\ \{u\}_k^{bot} \end{Bmatrix} \tag{15}$$

Matrices $[D]^{top}$, $[D]^{bot}$ contain the coefficients associated with stresses and matrices $[P_S]^{top}$, $[P_S]^{bot}$ representing the coefficients associated with displacements. $[H]^{top}$, $[H]^{bot}$ denote the diagonal matrix elements in (2). Finally, matrices $[A]_k$ and $[B]_k$ take the following forms [27]:

$$[A]_k = \begin{bmatrix} D_{11} & D_{13} & D_{15} & D_{11}e^{i\xi\alpha_1 d_k} & D_{13}e^{i\xi\alpha_3 d_k} & D_{15}e^{i\xi\alpha_5 d_k} \\ D_{21} & D_{23} & D_{25} & -D_{21}e^{i\xi\alpha_1 d_k} & -D_{23}e^{i\xi\alpha_2 d_k} & -D_{25}e^{i\xi\alpha_5 d_k} \\ D_{31} & D_{33} & D_{35} & -D_{31}e^{i\xi\alpha_1 d_k} & -D_{33}e^{i\xi\alpha_2 d_k} & -D_{35}e^{i\xi\alpha_5 d_k} \\ D_{11}e^{i\xi\alpha_1 d_k} & D_{13}e^{i\xi\alpha_3 d_k} & D_{15}e^{i\xi\alpha_5 d_k} & D_{11} & D_{13} & D_{15} \\ D_{21}e^{i\xi\alpha_1 d_k} & D_{23}e^{i\xi\alpha_3 d_k} & D_{25}e^{i\xi\alpha_5 d_k} & D_{21} & D_{23} & D_{25} \\ D_{31}e^{i\xi\alpha_1 d_k} & D_{33}e^{i\xi\alpha_3 d_k} & D_{35}e^{i\xi\alpha_5 d_k} & D_{31} & D_{33} & D_{35} \end{bmatrix} \tag{16}$$

$$[B]_k = \begin{bmatrix} 1 & 1 & 1 & e^{i\xi\alpha_1 d_k} & e^{i\xi\alpha_3 d_k} & e^{i\xi\alpha_5 d_k} \\ V_1 & V_3 & V_5 & V_1 e^{i\xi\alpha_1 d_k} & V_3 e^{i\xi\alpha_2 d_k} & V_5 e^{i\xi\alpha_5 d_k} \\ W_1 & W_3 & W_5 & -W_1 e^{i\xi\alpha_1 d_k} & -W_3 e^{i\xi\alpha_2 d_k} & -W_5 e^{i\xi\alpha_5 d_k} \\ e^{i\xi\alpha_1 d_k} & e^{i\xi\alpha_3 d_k} & e^{i\xi\alpha_5 d_k} & 1 & 1 & 1 \\ V_1 e^{i\xi\alpha_1 d_k} & V_3 e^{i\xi\alpha_3 d_k} & V_5 e^{i\xi\alpha_5 d_k} & V_1 & V_3 & V_5 \\ W_1 e^{i\xi\alpha_1 d_k} & W_3 e^{i\xi\alpha_3 d_k} & W_5 e^{i\xi\alpha_5 d_k} & -W_1 & -W_3 & -W_5 \end{bmatrix} \tag{17}$$

In order to obtain the stiffness matrix for the whole composite material, an advanced recursive algorithm has to be applied [24]. Let us consider two adjoining layers (1, 2), namely:

$$\begin{Bmatrix}\{\sigma\}_0\\ \{\sigma\}_1\end{Bmatrix}=\begin{bmatrix}[K]^A_{11} & [K]^A_{12}\\ [K]^A_{21} & [K]^A_{22}\end{bmatrix}\begin{Bmatrix}\{u\}_0\\ \{u\}_1\end{Bmatrix},$$
$$\begin{Bmatrix}\{\sigma\}_1\\ \{\sigma\}_2\end{Bmatrix}=\begin{bmatrix}[K]^B_{11} & [K]^B_{12}\\ [K]^B_{21} & [K]^B_{22}\end{bmatrix}\begin{Bmatrix}\{u\}_1\\ \{u\}_2\end{Bmatrix} \tag{18}$$

where the subscripts denote the interfaces. By excluding $\{\sigma\}_1$ and $\{u\}_1$ from the first relation and substituting it in the second one, the matrix which relates $\{\sigma\}_0$ $\{u\}_0$ to $\{\sigma\}_2$ $\{u\}_2$, is obtained. This combined matrix is an SM for these two bonded layers, namely:

$$\begin{Bmatrix}\{\sigma\}_0\\ \{\sigma\}_2\end{Bmatrix}=$$
$$\begin{bmatrix}[K]^A_{11}+[K]^A_{12}\left([K]^B_{11}-[K]^A_{22}\right)^{-1}[K]^A_{21} & -[K]^A_{12}\left([K]^B_{11}-[K]^A_{22}\right)^{-1}[K]^B_{12}\\ [K]^B_{21}\left([K]^B_{11}-[K]^A_{22}\right)^{-1}[K]^A_{21} & [K]^B_{22}-[K]^B_{21}\left([K]^B_{11}-[K]^A_{22}\right)^{-1}[K]^B_{12}\end{bmatrix}\begin{Bmatrix}\{u\}_0\\ \{u\}_2\end{Bmatrix}. \tag{19}$$

Denoting the SM obtained by $[K]^A$ and the SM for the third layer by $[K] = [K]^B$, we can recursively apply relation (19) to obtain the global stiffness matrix, which relates the stresses to the displacement for the top and bottom surface of the whole composite plate. It is worth stressing here that [24]: "submatrices $[K]^B{}_{11}$ and $[K]^A{}_{22}$ will approach semi-space stiffness with a thickness increase. Therefore, the inverse matrix in (19), $([K]^B{}_{11} - [K]^A{}_{22})^{-1}$, will always be regular. Thus, the recursive algorithm maintains the regularity of the layer stiffness matrix." The wave characteristic equation for the whole composite structure is obtained from the total stiffness matrix. Assuming that the stress components on the top and bottom surfaces are equal to zero, the Lamb wave dispersion equation is:

$$\det([K]) = 0. \tag{20}$$

Finally, in order to find the solution of (20), the determinant of the 6x6 matrix has to be computed. In contrast to TMM, the solution to this problem requires the use of a suitable numerical procedure.

SEMI-ANALYTICAL FINITE ELEMENT METHOD

The above discussed methods should be considered as analytical methods and they can be used only in the case of simple structures like plates or cylinders. In the case of structures of an arbitrary shape, the numerical method can be applied. SAFE is the method based on the finite element approach. The groundwork of this algorithm was formulated in 1971 by Nelson and coauthors [28]. The FE approach method has been successfully applied by numerous authors in the case of isotropic thin-walled structures, beams with different cross sections [29-32] or anisotropic shells and rods [33]. The main advantage of this approach is that in order to determine dispersion curves, commercial FE systems can be used, for example ANSYS [34] or ABAQUS [35]. The short description of the fundamentals of this method is taken from Sorohan et al. [36] and Kalkowski [37]. The general motion equation of a nongyroscopic finite element model can be written as:

$$[M]\{\ddot{u}\}+[C]\{\dot{u}\}+[K]\{u\}=\{F\}, \tag{21}$$

where [M] is the structural mass matrix, [C] is the structural viscous damping matrix and [K] is the structural stiffness matrix. {F} is the vector of applied loads. {u} is the displacement vector and its time derivatives, respectively. In further analysis, it is assumed that damping is neglected, i.e. [C] = 0. The investigated structure is considered as a set of identical, periodic elements as shown in Figure 1.

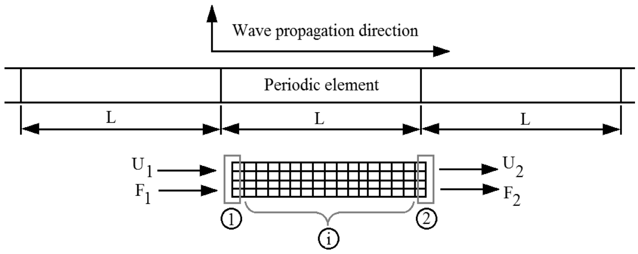

Fig. 1. Several coupled periodic elements and FE mesh [36]

Rys. 1. Kilka elementów periodycznych oraz siatka MES [36]

It should be emphasized here that the number and coordinates of nodes on the left and right boundaries have to be identical. Hence, the number of degrees of freedom (DOF) on the left and right boundaries are $n_1 = n_2 = n$, whereas the total number of DOF inside the periodic element is denoted as n_j. Furthermore, it is assumed that the displacement field is in the following form:

$$(u_1,u_2,u_3)=(U_1,U_2,U_3)e^{i(\xi x-\omega t)}, \tag{22}$$

where ξ is the wave number and ω is the circular frequency. When the periodic element vibrates harmonically with circular frequency ω, motion equation (21) takes the following form [39]:

$$\left(\begin{bmatrix}[K_{11}] & [K_{1j}] & [K_{12}]\\ [K_{1j}]^T & [K_{jj}] & [K_{j2}]\\ [K_{12}]^T & [K_{j2}]^T & [K_{22}]\end{bmatrix}-\omega^2\begin{bmatrix}[M_{11}] & [M_{1j}] & [M_{12}]\\ [M_{1j}]^T & [M_{jj}] & [M_{j2}]\\ [M_{12}]^T & [M_{j2}]^T & [M_{22}]\end{bmatrix}\right)\begin{Bmatrix}\{U_1\}\\ \{U_j\}\\ \{U_2\}\end{Bmatrix}=\begin{Bmatrix}\{F_1\}\\ \{F_j\}\\ \{F_2\}\end{Bmatrix}, \tag{23}$$

where the symmetry of appropriate matrices are outlined. When a free wave travels through an infinite structure, then $\{F_j\} = \{0\}$. However, the nodal forces on boundaries $\{F_1\}$, $\{F_2\}$ are not equal to zero. These forces are responsible for transmitting the wave motion from one periodic element to another. Bloch's theorem [38] says that the ratio between the corresponding displacement in adjacent periodic elements is equal to e^{μ}, where $\mu = i\xi L$ and L is the length of the periodic element (Fig. 1) [39]. Thus the nodal displacement and nodal forces on boundary 1 and 2 are related as follows:

$$\{U_2\} = e^{i\xi L}\{U_1\}, \quad \{F_2\} = -e^{i\xi L}\{F_1\}. \tag{24}$$

Next, it is assumed that $e^{i\xi L} = 1$. Hence $\xi L = \pm 2p\pi$, $p = 0, 1, 2, \ldots \infty$. Now, relations (24) become very simple , namely:

$$\{U_2\} = \{U_1\}, \quad \{F_2\} = -\{F_1\}. \tag{25}$$

Thus, the nodal displacement on boundaries 1 and 2 can be rewritten in matrix form as follows:

$$\begin{Bmatrix} \{U_1\} \\ \{U_j\} \\ \{U_2\} \end{Bmatrix} = [Q]\begin{Bmatrix} \{U_1\} \\ \{U_j\} \end{Bmatrix}, \quad [Q] = \begin{bmatrix} [I] & 0 \\ 0 & [I] \\ [I] & 0 \end{bmatrix}, \tag{26}$$

where $[I]$ is the identity matrix and $[Q]$ is the coupling matrix. Substituting (26) into (23) and under the condition that $\{F_2\} = -\{F_1\}$, the following eigenvalue problem is obtained:

$$\left(\begin{bmatrix} [K_{11}]+2[K_{12}]+[K_{22}] & [K_{j1}]+[K_{j2}] \\ [K_{j1}]+[K_{j2}] & [K_{jj}] \end{bmatrix} - \omega^2 \begin{bmatrix} [M_{11}]+2[M_{12}]+[M_{22}] & [M_{j1}]+[M_{j2}] \\ [M_{j1}]+[M_{j2}] & [M_{jj}] \end{bmatrix}\right)\begin{Bmatrix} \{U_1\} \\ \{U_j\} \end{Bmatrix} = \{0\}. \tag{27}$$

The solution of the above problem has $N = n + n_j$ eigenvalues ω_p^2 and associated eigenvectors φ_p. According to [36], these eigenvectors are orthogonal with respect to the mass and stiffness matrices. Usually, they are arranged in ascending order, namely: $\omega_1^2 \le \omega_2^2 \le \omega_3^2 \le \ldots \omega_N^2$. Keeping in mind that the wave number is equal to $\xi = 2\pi/\lambda$ and $\xi L = \pm 2p\pi$, the lengths of the traveling waves, described by eigenvectors φ_p, are:

$$\lambda_p = \frac{L}{p}, \quad p = 0,1,2,\ldots,\infty. \tag{28}$$

The corresponding following phase velocities can be determined as:

$$c_p = \lambda_p f_p, \quad f_p = \frac{\omega_p}{2\pi} \tag{29}$$

whereas the group velocities can be estimated with the use of finite difference schema. Finally, it is worth stressing here that the case when $p = 0$, $\lambda = \infty$, corresponds to the case of cut on frequencies [36]. In the work by Sorohan et al. [36] other examples are also presented, namely: isotropic plate, isotropic circular and square pipe and layered composite plate. In the all the studied cases, the obtained solutions agree with the results, which are obtained with the use of a different method. Nevertheless, in the case of a three-dimensional FE model of a circular or square pipe, the number of solid elements is very large. Hence, the numerical analysis takes a great deal of time and thus its efficiency is not obvious.

NUMERICAL EXAMPLE OF DETERMINING DISPERSION CURVES

Exemplary dispersion curves are determined for the 8-th layer composite plate, which is made of carbon fiber/epoxy resin, namely Fibers T300, Matrix N5208. The total thickness of the plate is equal to d = 2 mm. The plate consists of 8 layers with following stacking sequence [0°, 90°, 0°, 90°, 0°, 90°, 0°, 90°]. The layers also have identical thickness d_k = 0.25 mm. The assumed mechanical properties of the layer are as follows: E_1 = 181 GPa, E_2 = 10.3 GPa, G_{12} = 7.17 GPa, ν_{12} = 0.28 and density ρ = 1.6 g/cm³. It is worth noting here that the carbon layers are strongly orthotropic. SMM is used in order to determine the dispersion curves. In order to find the solution of the studied problem, an appropriate computer program is developed with the aid of SCILAB free software. In order to find the solution of the wave characteristic equation the bisection method is applied. The numerical algorithm of determining dispersion curves is adopted from the work by Lowe [1]. Although this work concerns composites where all the plies have isotropic mechanical properties, the algorithm described there can also be used in the case of arbitrary composite materials. The obtained results are depicted in Figure 2. It is assumed that the wave front propagates with angle φ = 45° with respect to the global coordinate system.

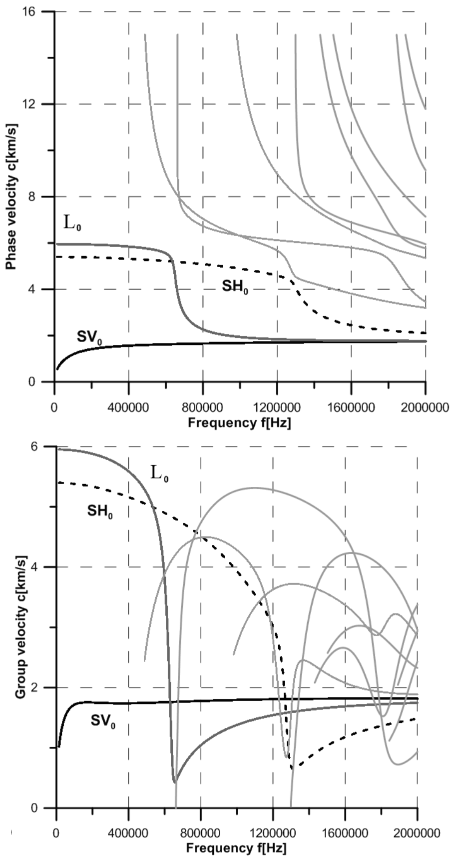

Fig. 2. Phase and group velocities. Layered composite [0°, 90°, 0°, 90°, 0°, 90°, 0°, 90°]. Wave propagation angle φ = 45°

Rys. 2. Prędkości fazowe i grupowe. Laminat [0°, 90°, 0°, 90°, 0°, 90°, 0°, 90°]. Kąt propagacji fali φ = 45°

The initial phase velocity for the L_0 wave mode is equal to c = 5.95 km/s. For the frequency, which is close to $f \approx 600$ kHz, the phase velocity suddenly decreases. For higher frequencies its value is almost constant and equal to $c \approx 1.75$ km/s. The behavior of the fundamental shear horizontal mode SH_0 is similar to the L_0 wave mode. Additionally, its initial velocity c = 5.4 km/s. For frequency $f > 1.2$ MHz, a sudden drop is visible, which is similar to the fundamental symmetric mode L_0 with frequency $f > 600$ kHz. It is also characteristic in the case of group velocities of L_0 and SH_0 modes. For the low frequency, the SV_0 wave mode is strongly dispersive, however, for the higher frequency its phase velocity is almost constant and its value is equal to $c \approx 1.84$ km/s. Qualitatively, the obtained dispersion curves are very similar to those which are presented in the first part of this review for aluminum alloy (Figs. 1 and 2). The number of higher modes is equal to eight. Nonetheless, it is not possible to distinguish whether they are the symmetric (L), shear horizontal (SH) or shear vertical (SV) wave mode.

FINAL REMARKS

In the second part of this review, three analytical approaches for determining wave dispersion curves are presented, namely the transfer matrix method, global matrix method and stiffness matrix method. Besides the mentioned methods, one approach, which is based on the finite element method, is discussed. It seems that the stiffness matrix method is the most effective. It is relatively simple and, what is the most important, it is numerically unconditionally stable. The semi-analytical method can be applied with the use of commercially available software based on the finite element method. Thus it can be used in order to confirm results obtained with the use of any other method.

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